Applicable Analysis and Discrete Mathematics, **2** (2008), 1–30.
Available electronically at **http://pefmath.etf.bg.ac.yu**

OSCILLATION THEOREMS FOR CERTAIN HIGHER ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Some new oscillation theorems for higher-order nonlinear functional differential equations of the form

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(a(t)\left(\frac{\mathrm{d}^n x(t)}{\mathrm{d}t^n}\right)^{\alpha}\right)+q(t)f\big(x\big(g(t)\big)\big)=0,\quad \alpha>0,$$

are established.

1. INTRODUCTION

This paper is concerned with the oscillatory behavior of the higher-order nonlinear functional differential equation

$$L_{2n}x(t)+q(t)f\big(x\big(g(t)\big)\big)=0, \tag{1.1}$$

where the differential operator, L_{2n}, is defined recursively by

$$(1.1)' \quad \begin{cases} L_0x=x,\\ L_ix=\dfrac{\mathrm{d}}{\mathrm{d}t}L_{i-1}x,\quad i=1,2,\dots,n-1,\\ L_jx=\dfrac{\mathrm{d}^{j-n}}{\mathrm{d}t^{j-n}}\left(a\left(\dfrac{\mathrm{d}}{\mathrm{d}t}L_{n-1}x\right)^{\alpha}\right)=\dfrac{\mathrm{d}^{j-n}}{\mathrm{d}t^{j-n}}L_nx,\quad j=n,n+1,\dots,2n.\end{cases}$$

Clearly

$$L_ix=\frac{\mathrm{d}}{\mathrm{d}t}L_{i-1}x,\quad i=1,2,\dots,n-1,n+1,\dots,2n,$$

2000 Mathematics Subject Classification. 34C10.
Keywords and Phrases. Functional differential equations, oscillation, nonoscillation, comparison.

and

$$L_n x = a\left(\frac{\mathrm{d}}{\mathrm{d}t} L_{n-1}x\right)^{\alpha}.$$

In what follows we assume that

(i) $a(t),\ q(t) \in C\big([t_0,\infty), \mathbb{R}^+ = (0,\infty)\big)$,

(ii) $g(t) \in C\big([t_0,\infty), \mathbb{R} = (-\infty,\infty)\big)$ and $\lim_{t\to\infty} g(t) = \infty$,

(iii) $f \in C(\mathbb{R},\mathbb{R})$ and $xf(x) > 0$ for $x \neq 0$,

(iv) α is the ratio of two positive odd integers.

Also we assume that

$$\int^{\infty} a^{-1/\alpha}(s)\,\mathrm{d}s = \infty. \tag{1.2}$$

By a solution of equation (1.1) we mean a function $x \in C^n([t_0,\infty),\mathbb{R})$ together with $a(x^{(n)})^{\alpha} \in C^n([t_0,\infty),\mathbb{R})$ which satisfies equation (1.1) for all $t \geq t_x \geq t_0 \geq 0$. Here we are concerned with proper solutions of equation (1.1), i.e. those solutions $x(t)$ which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for every $T \geq t_x$. Such a solution is said to be *oscillatory* if it has an infinite sequence of zeros clustering at infinity and *nonoscillatory* if it has at most a finite number of zeros in its interval of existence. Equation (1.1) is called *oscillatory* if all its solutions are oscillatory.

The problem of obtaining the nonoscillation and oscillation of certain higher-order nonlinear functional differential equations of type (1.1) when $\alpha = 1$ and/or $\alpha > 0$ has been studied by a number of authors, see [**1**–**14**, **16**–**21**] and the references cited therein. Indeed, Mahfoud [**16**, **17**] discussed the oscillation of the special case of (1.1)

$$x^{(n)}(t) + a(t)f\big(x\big(q(t)\big)\big) = 0.$$

Our main objective in this paper is to present an asymptotic study on the oscillation of equation (1.1) and to establish some new oscillation criteria.

In Section 2 we give the proofs of some important lemmas which are useful throughout this paper. Section 3 is devoted to the study of equation (1.1) when f satisfies either $f^{(1/\alpha)-1}(x)f'(x) \geq k > 0$ for $x \neq 0$ or $f(x)\operatorname{sgn} x \geq |x|^{\alpha}$. Also, our results involve comparison with related linear and half-linear second-order differential equations. In Section 4 we present some sufficient conditions for the oscillation of equation (1.1) when f satisfies either the condition $\int^{\pm\infty} \mathrm{d}u/f^{1/\alpha}(u) < \infty$ or the condition $\int_{\pm 0} \mathrm{d}u/f(u^{1/\alpha}) < \infty$. Section 5 is devoted to study of some necessary and sufficient conditions for the oscillation of equation (1.1). In Section 6 we give a comparison result which allows us to extend the results obtained to functional differential equations of neutral type and to equations of type (1.1) when the function f need not be monotonic. The results obtained extend, improve and corollate a number of existing results.

2. PRELIMINARIES

To obtain our main results we need the following lemma which is a generalization of the well-known lemma of Kiguradze [**3**].

Lemma 2.1. *Let $x(t)$ be a nonoscillatory solution of equation* (1.1) *and condition* (1.2) *hold. Then there exist an odd integer $k \in \{1, 3, \ldots, 2n-1\}$ and a $T \geq t_0$ such that for $t \geq T$,*

$$\left\{\begin{array}{l} x(t)L_i x(t) > 0 \text{ for } i = 0, 1, \ldots, k-1 \text{ and} \\ (-1)^{i+k} x(t) L_i x(t) > 0 \text{ for } i = k, k+1, \ldots, 2n-1. \end{array}\right. \tag{2.1}$$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0$. Since $L_{2n}x(t) \leq 0$ for $t \geq t_0$, it follows that $L_i x(t)$, $i = 1, 2, \ldots, 2n-1$, are eventually of constant sign. Firstly we prove that $L_{2n-1}x(t) > 0$ for $t \geq T$ for some $T \geq t_0$. To this end, we suppose that for some $T_1 \geq T$ we have $L_{2n-1}x(t) \leq 0$, $t \geq T_1$. Then, since $L_{2n-1}x(t)$ is decreasing and not identically zero on $[T_1, \infty)$, there exist a $T_2 \geq T_1$ and a constant $c > 0$ such that $L_i x(t) \leq -c$ for $t \geq T_2$ and $i \in \{1, 3, \ldots, 2n-1\}$ for otherwise integration of the inequality would imply that $L_0 x(\infty) = x(\infty) = -\infty$, which contradicts the fact that $x(t) > 0$ on $[t_0, \infty)$. From this fact it follows that none of the consecutive derivatives $L_i x(t)$ and $L_{i+1}x(t)$ can be eventually negative.

Next the positivity of $L_{2n-1}x(t)$ on $[T, \infty)$ implies that $L_{2n-2}x(t)$ is increasing there. Here there are two cases to consider:

Case (I). $L_{2n-2}x(t) > 0$ on $[t_1, \infty)$ for some $t_1 \geq T$. There exist a constant $c_1 > 0$ and a $t_2 \geq t_1$ such that $L_{2n-2}x(t) \geq c_1$ for $t \geq t_2$. One can easily see that $L_i x(\infty) = \infty$ for $i = 1, 2, \ldots, 2n-3$ which shows that $L_i x(t)$, $i = 1, 2, \ldots, 2n-3$, are eventually positive.

Case (II). $L_{2n-2}x(t) < 0$ on $[\overline{T}, \infty), \overline{T} \geq T$. Clearly $L_{2n-3}x(t)$ must remain positive on $[\overline{T}, \infty)$ since the simultaneous negativity of $L_{2n-2}x(t)$ and $L_{2n-3}x(t)$ is impossible.

Repeatedly applying the same arguments as above we arrive at the desired conclusion. □

From Lemma 2.1 we distinguish the following three cases: (i) $k = 2n-1$, (ii) $n+1 \leq k \leq 2n-3$ and (iii) $1 \leq k \leq n$.

(i) Let $k = 2n-1$. Since $L_{2n-1}x(t) > 0$ is decreasing on $[T, \infty)$, we have

$$L_{2n-2}x(t) \geq (t-T)L_{2n-1}x(t), \quad t \geq T.$$

Integrating this inequality $(n-2)$ times from T to t and using the decreasing property of $L_{2n-1}x(t)$, we obtain

$$L_n x(t) \geq \frac{(t-T)^{n-1}}{(n-1)!} L_{2n-1}x(t), \quad t \geq T$$

or

$$x^{(n)}(t) \ge \left(\frac{(t-T)^{n-1}}{(n-1)!\,a(t)}\right)^{1/\alpha} L_{2n-1}^{1/\alpha} x(t), \quad t \ge T.$$

By applying TAYLOR's formula with integral remainder we get

$$x^{(j)}(t) \ge \left(\int_T^t \frac{(t-u)^{n-j-1}}{(n-j-1)!} \left(\frac{(u-T)^{n-1}}{(n-1)!a(u)}\right)^{1/\alpha} \mathrm{d}u\right) L_{2n-1}^{1/\alpha} x(t) \tag{2.2}$$

for $j = 0, 1, \ldots, n-1$ and $t \ge T$.

(ii) Let $n+1 \le k \le 2n-3$. From Lemma 2.1 we see that $L_{2n-1}x(t) > 0$ is decreasing and $L_{2n-2}x(t) < 0$ for $t \ge T$. Now

$$L_{2n-2}x(t) - L_{2n-2}x(s) = \int_s^t L_{2n-1}x(u)\,\mathrm{d}u, \quad t \ge s \ge T$$

and so

$$-L_{2n-2}x(s) \ge (t-s)L_{2n-1}x(t). \tag{2.3}$$

Integrating the above inequality $(2n-k-2)$ times from s to t yields

$$(-1)^{2n-k-1} L_k x(s) \ge \frac{(t-s)^{2n-k-1}}{(2n-k-1)!} L_{2n-1}x(t)$$

or

$$L_k x(s) \ge \frac{(t-s)^{2n-k-1}}{(2n-k-1)!} L_{2n-1}x(t) \quad \text{for} \quad t \ge s \ge T. \tag{2.4}$$

Integrating (2.4) $(k-n)$ times from T to $s\,(\ge T)$ we have

$$L_n x(s) \ge \left(\int_T^s \frac{(s-u)^{k-n-1}}{(k-n-1)!} \frac{(t-u)^{2n-k-1}}{(2n-k-1)!} \mathrm{d}u\right) L_{2n-1}x(t)$$

or, equivalently,

$$x^{(n)}(s) \ge \left(\frac{1}{a(s)} \int_T^s \frac{(s-u)^{k-n-1}}{(k-n-1)!} \frac{(t-u)^{2n-k-1}}{(2n-k-1)!} \mathrm{d}u\right)^{1/\alpha} L_{2n-1}^{1/\alpha} x(t).$$

As in case (i) one can easily find that

$$\begin{aligned} x^{(j)}(s) & \\ \ge & \left(\int_T^s \frac{(s-v)^{n-j-1}}{(n-j-1)!} \left(\frac{1}{a(v)} \int_T^v \frac{(v-u)^{k-n-1}}{(k-n-1)!} \frac{(t-u)^{2n-k-1}}{(2n-k-1)!} \mathrm{d}u\right)^{1/\alpha} \mathrm{d}v\right) L_{2n-1}x(t) \end{aligned} \tag{2.5}$$

for $j = 0, 1, \ldots, n-2,\ t \ge s \ge T$.

(iii) Let $1 \le k \le n$. Then as in case (ii), we obtain (2.3) for $t \ge s \ge T$. Integrating (2.3) $(n-2)$ times from s to $t (\ge s \ge T)$ one can easily find that

$$(2.6) \qquad (-1)^{n-1} x^{(n)}(s) \ge \left(\frac{1}{a(s)} \frac{(t-s)^{n-1}}{(n-1)!} \right)^{1/\alpha} L_{2n-1} x(t).$$

Next integrating (2.6) $(n-k)$ times from s to $t\,(\ge s \ge T)$ we find that

$$\begin{aligned} &(-1)^{2n-k-1} x^{(k)}(s) \\ &\quad = x^{(k)}(s) \ge \left(\int_s^t \frac{(u-s)^{n-k-1}}{(n-k-1)!} \left(\frac{1}{a(u)} \frac{(t-u)^{n-1}}{(n-1)!} \right)^{1/\alpha} \mathrm{d}u \right) L_{2n-1}^{1/\alpha} x(t). \end{aligned}$$

As in case (i) we find that

$$\begin{aligned} &x^{(j)}(s) \\ &\ge \left(\int_T^s \frac{(s-v)^{k-j-1}}{(k-j-1)!} \left(\int_v^t \frac{(u-v)^{n-k-1}}{(n-k-1)!} \left(\frac{1}{a(u)} \frac{(t-u)^{n-1}}{(n-1)!} \right)^{1/\alpha} \mathrm{d}u \right) \mathrm{d}v \right) L_{2n-1}^{1/\alpha} x(t) \end{aligned}$$

for $k-1 \ge j = 0, 1$, and, if $k - 1 < j = 0, 1$,

$$x^{(j)}(s) \ge \left(\int_s^t \frac{(u-s)^{n-2}}{(n-2)!} \left(\frac{1}{a(u)} \frac{(t-u)^{n-1}}{(n-1)!} \right)^{1/\alpha} \mathrm{d}u \right) L_{2n-1}^{1/\alpha} x(t).$$

For $t \ge T/\lambda \ge t_0,\ k \in \{1, 3, \ldots, 2n-1\}$ and for some constant $\lambda,\ 0 < \lambda < 1$, we define

$$\begin{aligned} H_j(t, T; a; k; \lambda) = \min \Bigg\{ &\int_T^{\lambda t} \frac{(\lambda t - u)^{n-j-1}}{(n-j-1)!} \left(\frac{(u-T)^{n-1}}{(n-1)!} \frac{1}{a(u)} \right)^{1/\alpha} \mathrm{d}u \\ &\qquad \text{if } k = 2n-1, \\ &\int_T^{\lambda t} \frac{(\lambda t - v)^{n-j-1}}{(n-j-1)!} \left(\frac{1}{a(v)} \int_T^v \frac{(v-u)^{k-n-1}}{(k-n-1)!} \frac{(t-u)^{2n-k-1}}{(2n-k-1)!} \mathrm{d}u \right)^{1/\alpha} \mathrm{d}v \\ &\qquad \text{if } n+1 \le k \le 2n-3, \\ &\int_T^{\lambda t} \frac{(\lambda t - v)^{k-j-1}}{(k-j-1)!} \left(\int_v^t \frac{(u-v)^{n-k-1}}{(n-k-1)!} \left(\frac{1}{a(u)} \frac{(t-u)^{n-1}}{(n-1)!} \right)^{1/\alpha} \mathrm{d}u \right) \mathrm{d}v \\ &\qquad \text{if } 1 \le k \le n, \quad k-1 \ge j,\ j = 0, 1, \\ &\int_{\lambda t}^t \frac{(u - \lambda t)^{n-2}}{(n-2)!} \left(\frac{1}{a(u)} \frac{(t-u)^{n-1}}{(n-1)!} \right)^{1/\alpha} \mathrm{d}u \text{ if } 1 \le k \le n,\ k-1 < j,\ j = 0, 1 \Bigg\}. \end{aligned}$$

We are now ready to state the following important lemma.

Lemma 2.2. *Let $x(t)$ be a positive solution of equation* (1.1). *Then for some constant λ, $0 < \lambda < 1$ and all large $t \geq T \geq t_0$ and for $k \in \{1, 3, \dots, 2n-1\}$,*

$$x'(\lambda t) \geq H_1(t, T; a; k; \lambda)\, L_{2n-1}^{1/\alpha} x(t) \tag{2.7}$$

and

$$x(t) \geq x(\lambda t) \geq H_0(t, T; a; k; \lambda)\, L_{2n-1}^{1/\alpha} x(t). \tag{2.8}$$

We shall also need the following lemmas.

Lemma 2.3 [**15**]. *If X and Y are nonnegative, then*

$$X^{\overline{\lambda}} + (\overline{\lambda} - 1)Y^{\overline{\lambda}} - \overline{\lambda} X Y^{\overline{\lambda}-1} \geq 0, \quad \overline{\lambda} > 1,$$

where equality holds if and only if $X = Y$.

Lemma 2.4 [**4, 5**]. *The semilinear differential equation*

$$\big(a(t)\big(x'(t)\big)^{\alpha}\big)' + q(t)x^{\alpha}(t) = 0, \tag{2.9}$$

where a, q and x are as in equation (1.1) *is nonoscillatory if and only if there exist a number $T \geq t_0$ and a function $v(t) \in C^1([t_0, \infty), \mathbb{R}$ which satisfies the inequality*

$$v'(t) + \alpha a^{-1/\alpha}(t)\, |v(t)|^{1+1/\alpha} + q(t) \leq 0 \ \textit{on} \ [T, \infty).$$

Lemma 2.5 [**5**]. *Let $h(t) \in C([T, \infty), \mathbb{R}^+)$, $T \geq t_0$. If there exists a function $v(t) \in C^1([T, \infty), \mathbb{R})$ such that*

$$v'(t) + h(t)v^2(t) + q(t) \leq 0 \ \textit{for every} \ t \geq T,$$

then the second-order linear differential equation

$$\left(\frac{1}{h(t)}\, x'(t)\right)' + q(t)x(t) = 0$$

is nonoscillatory.

3. OSCILLATION AND COMPARISON RESULTS

In this section we present some sufficient conditions for the oscillation of equation (1.1). Also our results involve comparison with related linear and semilinear second-order differential equations so that the known oscillation theorems from the literature can be employed directly.

In what follows we assume that

$$f^{1/\alpha-1}(x)f'(x) \geq \overline{k} > 0 \text{ for } x \neq 0 \text{ and } \overline{k} \text{ is a constant.} \tag{3.1}$$

We also assume that there exists a function, $\sigma(t) \in C^1([t_0,\infty),\mathbb{R}^+)$, such that

$$\sigma(t) \leq \inf\{t,\ g(t)\},\quad \sigma'(t) > 0 \text{ for } t \geq t_0 \text{ and } \lim_{t\to\infty} \sigma(t) = \infty. \tag{3.2}$$

Theorem 3.1. *Let conditions* (1.2), (3.1) *and* (3.2) *hold. If there exist a function* $\rho(t) \in C^1([t_0,\infty),\mathbb{R}^+)$ *and a constant* λ, $0 < \lambda < 1$, *such that for* $\sigma(t) > T/\lambda$, $T \geq t_0$, *then*

$$\limsup_{t\to\infty} \int_T^t \Bigg(\rho(s)q(s) - \frac{1}{(\lambda\overline{k})^\alpha}\,\frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\,\frac{(\rho'(s))^{\alpha+1}}{\big(\rho(t)\sigma'(t)H_1(\sigma(s),T;a;k;\lambda)\big)^\alpha}\Bigg)\mathrm{d}s = \infty, \tag{3.3}$$

where H_1 *is as in Lemma* 2.2, $k \in \{1,3,\dots,2n-1\}$. *Then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. From equation (1.1) we see that $L_{2n}x(t) \leq 0$ for $t \geq t_0$ and so $L_ix(t)$, $i = 1,2,\dots,2n$, are eventually of one sign. By Lemma 2.1 there exists a $t_1 \geq t_0$ and $k \in \{1,3,\dots,2n-1\}$ such that (2.1) holds for $t \geq t_1$. By applying Lemma 2.2 there exist a $T \geq t_1$ and a $\lambda, 0 < \lambda < 1$, such that for all large $t \geq \sigma(t) > T/\lambda$

$$x'\big(\lambda\sigma(t)\big) \geq H_1(\sigma(t),T;a;k;\lambda)\,L_{2n-1}^{1/\alpha}x(t). \tag{3.4}$$

Define

$$w(t) = \rho(t)\,\frac{L_{2n-1}x(t)}{f\big(x\big(\lambda\sigma(t)\big)\big)} \quad \text{for} \quad t \geq T. \tag{3.5}$$

Then for $t \geq T$ we have

$$\begin{aligned} w'(t) &= \rho(t)\,\frac{\big(L_{2n-1}x(t)\big)'}{f\big(x\big(\lambda\sigma(t)\big)\big)} + \rho'(t)\,\frac{L_{2n-1}x(t)}{f\big(x\big(\lambda\sigma(t)\big)\big)} \\ &\qquad - \lambda\rho(t)\sigma'(t)\,\frac{f'\big(x\big(\lambda\sigma(t)\big)\big)}{f^{1-1/\alpha}\big(x\big(\lambda\sigma(t)\big)\big)}\,\frac{L_{2n-1}x(t)x'\big(\lambda\sigma(t)\big)}{f^{1+1/\alpha}\big(x\big(\lambda\sigma(t)\big)\big)} \\ &= -\rho(t)q(t)\,\frac{f\big(x\big(g(t)\big)\big)}{f\big(x\big(\lambda\sigma(t)\big)\big)} + \frac{\rho'(t)}{\rho(t)}\,w(t) \\ &\qquad - \lambda\rho(t)\sigma'(t)\,\frac{f'\big(x\big(\lambda\sigma(t)\big)\big)}{f^{1-1/\alpha}\big(x\big(\lambda\sigma(t)\big)\big)}\,\frac{L_{2n-1}x(t)x'\big(\lambda\sigma(t)\big)}{f^{1+1/\alpha}\big(x\big(\lambda\sigma(t)\big)\big)}\,. \end{aligned} \tag{3.6}$$

Using (3.1) and (3.4) in (3.6) we obtain

$$(3.7)\qquad w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}\, w(t) - \lambda\overline{k}\rho^{-1/\alpha}(t)\sigma'(t)H_1\big(\sigma(t),T;a;k;\lambda\big)w^{1+1/\alpha}(t) \quad \text{for} \quad t \ge T.$$

Setting

$$X = \Big(\lambda\overline{k}\rho^{-1/\alpha}(t)\sigma'(t)H_1\big(\sigma(t),T;a;k;\lambda\big)\Big)^{\alpha/(\alpha+1)} w(t), \quad \overline{\lambda} = \frac{\alpha+1}{\alpha} > 1$$

and

$$Y = \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha}\Big(\frac{\rho'(t)}{\rho(t)}\Big)^{\alpha}\Big(\big(\lambda\overline{k}\rho^{-1/\alpha}(t)\sigma'(t)H_1(\sigma(t),T;a;k;\lambda)\big)^{-\alpha/(\alpha+1)}\Big)^{\alpha}$$

in Lemma 2.3 we conclude that

$$\frac{\rho'(t)}{\rho(t)}\, w(t) - \lambda k\rho^{-1/\alpha}(t)\sigma'(t)H_1\big(\sigma(t),T;a;k;\lambda\big)w^{1+1/\alpha}(t) \le \frac{1}{(\lambda\overline{k})^{\alpha}}\,\frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}\,\frac{\big(\rho'(t)\big)^{\alpha+1}}{\big(\rho(t)\sigma'(t)H_1\big(\sigma(t),T;a;k;\lambda\big)\big)^{\alpha}} \quad \text{for} \quad t \ge T.$$

Thus it follows from (3.7) that

$$w'(t) \le -\rho(t)q(t) + \frac{1}{(\lambda\overline{k})^{\alpha}}\,\frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}\,\frac{\big(\rho'(t)\big)^{\alpha+1}}{\big(\rho(t)\sigma'(t)H_1\big(\sigma(t),T;a;k;\lambda\big)\big)^{\alpha}}, \quad t \ge T.$$

Integrating the above inequality from T to t we have

$$(3.8)\qquad 0 < w(t) \le w(T) - \int_T^t \Big(\rho(s)q(s) - \frac{1}{(\lambda\overline{k})^{\alpha}}\,\frac{\alpha^{\alpha}}{(1+\alpha)^{1+\alpha}}\,\frac{\big(\rho'(s)\big)^{\alpha+1}}{\big(\rho(s)\sigma'(s)H_1\big(\sigma(s),T;a;k;\lambda\big)\big)^{\alpha}}\Big)\mathrm{d}s.$$

Taking lim sup of both sides of (3.8) as $t \to \infty$ and using condition (3.3) we find that $w(t) \to -\infty$ as $t \to \infty$, which is a contradiction. This completes the proof. □

Next we relate the oscillation of equation (1.1) to that of semilinear equations of type (2.9).

Theorem 3.2. *Let conditions* (1.2), (3.1) *and* (3.2) *hold. Suppose the semilinear second-order equation*

$$(3.9)\qquad \big(c(t)\big(y'(t)\big)^{\alpha}\big)' + q(t)\,y^{\alpha}(t) = 0$$

is oscillatory, where

$$c(t) = \Big(\frac{\lambda\overline{k}}{\alpha}\,\sigma'(t)H_1\big(\sigma(t),T;a;k;\lambda\big)\Big)^{-\alpha}.$$

Then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Proceed as in the proof of Theorem 3.1 with $\rho(t) = 1$ to obtain (3.8) which takes the form

$$w'(t) \leq -\rho(t)q(t) - \lambda\overline{k}\sigma'(t)H_1\big(\sigma(t), T; a; k; \lambda\big)w^{1+1/\alpha}(t) \quad \text{for} \quad t \geq T.$$

Applying Lemma 2.4 to the above inequality we conclude that the equation (3.9) is nonoscillatory, which is a contradiction and completes the proof. □

Theorem 3.3. *Let* $\alpha \geq 1$, *conditions* (1.2) *and* (3.2) *hold and*

$$f(x)\operatorname{sgn} x \geq |x|^\beta \quad \textit{for} \quad x \neq 0, \tag{3.10}$$

where β *is the ratio of two positive odd integers. If there exist a function* $\rho(t) \in C^1([t_0,\infty), \mathbb{R}^+)$ *and a constant* λ, $0 < \lambda < 1$ *such that for* $\sigma(t) > T/\lambda$, $T \geq t_0$,

$$\limsup_{t\to\infty} \int_T^t \Bigg(\rho(s)q(s) - \frac{\big(\rho'(s)\big)^2}{4\lambda\beta\sigma'(s)\rho(s)\eta(s)H_1\big(\sigma(s), T; a; k; \lambda\big)H_0^{\alpha-1}\big(\sigma(s), T; a; k; \lambda\big)}\Bigg)\mathrm{d}s = \infty, \tag{3.11}$$

where H_i, $i = 0, 1$, *are as in Lemma* 2.2, $k \in \{1, 3, \dots, 2n-1\}$ *and*

$$\eta(t) = \begin{cases} c_1, \quad c_1 \text{ is any positive constant}, & \text{when} \quad \beta > \alpha, \\ 1, & \text{when} \quad \beta = \alpha, \\ c_2\,\phi^{\beta-\alpha}(t, t_0, a), \quad c_2 \text{ is any positive constant}, & \text{when} \quad \beta < \alpha, \end{cases} \tag{3.12}$$

with

$$\phi(t, t_0, a) = \int_{t_0}^t (t-s)^{n-1}\left(\frac{s^{n-1}}{a(s)}\right)^{1/\alpha}\mathrm{d}s, \tag{3.13}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 > 0$. Proceeding as in the proof of Theorem 3.1 we obtain (3.4) and also

$$x(t) \geq x\big(\sigma(t)\big) \geq x\big(\lambda\sigma(t)\big) \geq H_0\big(\sigma(t), T; a; k; \lambda\big)L_{2n-1}^{1/\alpha}x(t), \quad t \geq T. \tag{3.14}$$

Next there exist a constant $b_1 > 0$ and $\overline{T}_1 \geq t_0$ such that $L_{2n-1}x(t) \leq b_1$ for $t \geq \overline{T}_1$. Integrating this inequality from $\overline{T}_1$ to t one can easily see that there exist a constant $b > 0$ and a $T_1 \geq \overline{T}_1$ such that

$$x\big(\lambda\sigma(t)\big) \leq x(t) \leq b\phi(t, \overline{T}_1; a) \quad \text{for} \quad t \geq T_1. \tag{3.15}$$

Defining the function $w(t)$ by (3.5) and proceeding as in the proof of Theorem 3.1 to obtain (3.6) with $f(x)$ replaced by x^β, we obtain

$$w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}\, w(t) - \lambda\beta\rho(t)\sigma'(t)\, \frac{L_{2n-1}x(t)x'\big(\lambda\sigma(t)\big)}{x^{\beta+1}\big(\lambda\sigma(t)\big)} \quad \text{for } \ t \ge T. \tag{3.16}$$

Using (3.4) and (3.14) in inequality (3.16) we find

$$\begin{aligned} w'(t) &\le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}\, w(t) \\ &-\lambda\beta\, \frac{\sigma'(t)}{\rho(t)}\, H_1\big(\sigma(t),T;a;k;\lambda\big) H_0^{\alpha-1}\big(\sigma(t),T;a;k;\lambda\big) x^{\beta-\alpha}\big(\lambda\sigma(t)\big) w^2(t), \quad t \ge T. \end{aligned} \tag{3.17}$$

Next we consider the following three cases:

Case 1. If $\beta > \alpha$, then there exist a constant γ_1 and a $T_2 \ge T$ such that

$$x\big(\lambda\sigma(t)\big) \ge \gamma_1 \quad \text{for } \ t \ge T_2. \tag{3.18}$$

Thus inequality (3.17) takes the form

$$\begin{aligned} w'(t) &\le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}\, w(t) \\ &- \lambda\beta\gamma_1^{\beta-\alpha}\, \frac{\sigma'(t)}{\rho(t)}\, H_1\big(\sigma(t),T;a;k;\lambda\big) H_0^{\alpha-1}\big(\sigma(t),T;a;k;\lambda\big) w^2(t), \quad t \ge T_2. \end{aligned} \tag{3.19}$$

Case 2. If $\beta = \alpha$, then inequality (3.17) becomes

$$\begin{aligned} w'(t) &\le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}\, w(t) \\ &- \lambda\beta\, \frac{\sigma'(t)}{\rho(t)}\, H_1\big(\sigma(t),T;a;k;\lambda\big) H_0^{\alpha-1}\big(\sigma(t),T;a;k;\lambda\big) w^2(t), \quad t \ge T. \end{aligned} \tag{3.20}$$

Case 3. If $\beta < \alpha$, then by (3.15) we get

$$x^{\beta-\alpha}\big(\lambda\sigma(t)\big) \ge \gamma_2 \phi^{\beta-\alpha}(t,\overline{T}_1;a), \quad \gamma_2 = b^{\beta-\alpha}, \quad t \ge T_1, \tag{3.21}$$

and inequality (3.17) becomes

$$\begin{aligned} w'(t) &\le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}\, w(t) \\ &-\lambda\beta\gamma_2\phi^{\beta-\alpha}(t,\overline{T};a) H_1\big(\sigma(t),T;a;k;\lambda\big) H_0^{\alpha-1}\big(\sigma(t),T;a;k;\lambda\big) w^2(t), \ t \ge T_1. \end{aligned} \tag{3.22}$$

Choose $T^* = \max\{T, T_1, T_2\}$ and combine the inequalities (3.19), (3.20) and (3.22) to obtain

$$(3.23)\quad w'(t) \le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}\, w(t)$$

$$-\lambda\beta\,\frac{\sigma'(t)}{\rho(t)}\,\eta(t)H_1\big(\sigma(t),T;a;k;\lambda\big)H_0^{\alpha-1}\big(\sigma(t),T;a;k;\lambda\big)w^2(t),\quad t\ge T^*$$

$$= -\rho(t)q(t) - \Bigg(\bigg(\lambda\beta\,\frac{\sigma'(t)}{\rho(t)}\eta(t)H_1\big(\sigma(t),T;a;k;\lambda\big)H_0^{\alpha-1}(\sigma(t),T;a;k;\lambda)\bigg)^{1/2} w(t)$$

$$-\frac{\rho'(t)}{2\rho(t)\Big(\lambda\beta\,\frac{\sigma'(t)}{\rho(t)}\,\eta(t)H_1\big(\sigma(t),T;a;k;\lambda\big)H_0^{\alpha-1}\big(\sigma(t),T;a;k;\lambda\big)\Big)^{1/2}}\Bigg)^2$$

$$+\frac{\big(\rho'(t)\big)^2}{4\lambda\beta\sigma'(t)\rho(t)\eta(t)H_1\big(\sigma(t),T;a;k;\lambda\big)H_0^{\alpha-1}\big(\sigma(t),T;a;k;\lambda\big)}$$

$$(3.24)\quad \le -\Bigg(\rho(t)q(t) - \frac{\big(\rho'(t)\big)^2}{4\lambda\beta\sigma'(t)\rho(t)\eta(t)H_1\big(\sigma(t),T;a;k;\lambda\big)H_0^{\alpha-1}\big(\sigma(t),T;a;k;\lambda\big)}\Bigg),\quad t\ge T^*.$$

Integrating (3.24) from T^* to t we have

$$0 < w(t) \le w(T^*) - \int_{T^*}^{t}\Bigg(\rho(s)q(s) - \frac{\big(\rho'(s)\big)^2}{4\lambda\beta\sigma'(s)\rho(s)\eta(s)H_1\big(\sigma(s),T;a;k;\lambda\big)H_0^{\alpha-1}\big(\sigma(s),T;a;k;\lambda\big)}\Bigg)\mathrm{d}s.$$

Taking lim sup of both sides of the above inequality as $t\to\infty$ and by condition (3.11) we see that $w(t)\to-\infty$ as $t\to\infty$, which is a contradiction and completes the proof. □

In the following result we compare the oscillation of equation (1.1) with that of linear second-order ordinary differential equation.

Theorem 3.4. *Let* $\alpha\ge 1$, *conditions* (1.2) *and* (3.2) *hold and* (3.10) *hold. Suppose the linear second-order equation*

$$(3.25)\qquad \Big(\frac{1}{r(t)}\,y'(t)\Big)' + q(t)y(t) = 0$$

is oscillatory, where $r(t) = \lambda\beta\sigma'(t)\eta(t)H_1\big(\sigma(t),T;a;k;\lambda\big)H_0^{\alpha-1}\big(\sigma(t),T;a;k;\lambda\big)$. *Then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t)>0$ for $t\ge t_0\ge 0$. Proceed as in the proof of Theorem 3.3 with $\rho(t)=1$ to obtain (3.23) which takes the form

$$w'(t) \le -q(t) - \lambda\beta\sigma'(t)\eta(t)H_1\big(\sigma(t),T;a;k;\lambda\big)H_0^{\alpha-1}\big(\sigma(t),T;a;k;\lambda\big)w^2(t),\quad t\ge T^*.$$

Applying Lemma 2.5 to the above inequality we find that the equation (3.25) is nonoscillatory, which is a contradiction. This completes the proof. □

Next we present the following oscillation result for equation (1.1) when $0 < \alpha \le 1$.

Theorem 3.5. *Let* $0 < \alpha \le 1$, *conditions* (1.2), (3.2) *and* (3.10) *hold. Moreover assume that there exist a function* $\rho(t) \in C^1\big([t_0,\infty),\mathbb{R}^+\big)$ *and a constant* λ, $0 < \lambda < 1$, *such that for* $\sigma(t) > T/\lambda$, $T \ge t_0$,

$$(3.26)\qquad \limsup_{t\to\infty} \int_T^t \left(\rho(s)q(s) - \frac{\big(\rho'(s)\big)^2 Q^{1-1/\alpha}(s)}{4\lambda\beta\sigma'(s)\xi(s)H_1\big(\sigma(s),T;a;k;\lambda\big)}\right)\mathrm{d}s = \infty,$$

where H_1 *is as in Lemma* 2.2, $k \in \{1,3,\dots,2n-1\}$ *and* $Q(t) = \int_t^\infty q(s)\,\mathrm{d}s$ *and*

$$(3.27)\qquad \xi(t) = \begin{cases} c_1,\ c_1 \text{ is any positive constant}, & \text{when } \beta > \alpha,\\ 1, & \text{when } \beta = \alpha,\\ c_2\phi^{\beta/\alpha-1}(t,t_0;a),\ c_2 \text{ is any positive constant}, & \text{when } \beta < \alpha, \end{cases}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \ge t_0 > 0$. Define the function $w(t)$ by (3.5) with $f(x) = x^\beta$ and proceed as in the proof of Theorems 3.1 and 3.3 to obtain (3.4), (3.14) – (3.16) for $t \ge T$. Using (3.4) in (3.16) one can easily find that

$$(3.28)\qquad \begin{aligned} w'(t) &\le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t)\\ &\quad - \lambda\beta\sigma'(t)\rho^{-1/\alpha}(t)w^2(t)w^{1/\alpha-1}(t)H_1\big(\sigma(t),T;a;k;\lambda\big)x^{\beta/\alpha-1}\big(\lambda\sigma(t)\big). \end{aligned}$$

It is easy to see that

$$(3.29)\qquad w(t) \ge \rho(t)Q(t) \quad \text{for} \quad t \ge T.$$

Using (3.29) in (3.28) we obtain

$$(3.30)\qquad \begin{aligned} w'(t) &\le -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}w(t)\\ &\quad - \frac{\lambda\beta\sigma'(t)}{\rho(t)}Q^{1/\alpha-1}(t)H_1\big(\sigma(t),T;a;k;\lambda\big)w^2(t)x^{\beta/\alpha-1}\big(\lambda\sigma(t)\big), \quad t \ge T. \end{aligned}$$

The rest of the proof is similar to that of Theorem 3.3 and hence is omitted. □

In the following result we relate the oscillation of equation (1.1) for $0 < \alpha \le 1$ with that of linear second-order equations.

Theorem 3.6. *Let* $0 < \alpha \le 1$, *conditions* (1.2), (3.2) *and* (3.10) *hold. Suppose the linear second-order equation*

$$\Big(\frac{1}{h(t)}\, z'(t)\Big)' + q(t)z(t) = 0 \tag{3.31}$$

is oscillatory, where $h(t) = \lambda\beta\sigma'(t)\xi(t)Q^{1/\alpha-1}(t)H_1\big(\sigma(t),T;a;k;\lambda\big)$ *and* $\xi(t)$ *is given by* (3.27). *Then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \ge t_0 > 0$. Proceeding as in the proof of Theorem 3.5 with $\rho(t) = 1$ to obtain the inequality (3.30) which takes the form

$$w'(t) \le -q(t) - \lambda\beta\sigma'(t)\xi(t)Q^{1/\alpha-1}(t)H_1\big(\sigma(t),T;a;k;\lambda\big)w^2(t), \quad t \ge T.$$

The rest of the proof is similar to that of Theorem 3.4 and hence is omitted. □

For each $t \ge t_0$ we let $g(t) \le t$ and define $\mu(t) = \sup\{s \ge t_0 : g(s) \le t\}$. Clearly $\mu(t) \ge t$ and $g \circ \mu(t) = t$. Now we are ready to prove the following result.

Theorem 3.7. *Let* $g(t) \le t$ *for* $t \ge t_0$ *and conditions* (1.2) *and* (3.10) *hold with* $\alpha = \beta$. *If for all large* $T \ge t_0, k \in \{1,3,\dots,2n-1\}$ *and some constant* $\lambda, \lambda \in (0,1)$,

$$\limsup_{t\to\infty} H_0^\alpha(t,T;a;k;\lambda) \int_{\mu(t)}^{\infty} q(s)\,\mathrm{d}s \; > 1, \tag{3.32}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \ge t_0 \ge 0$. Integrating equation (1.1) from $t(\ge t_0)$ to $u(\ge t)$ and letting $u \to \infty$ we obtain

$$L_{2n-1}x(t) \ge \int_t^\infty q(s)x^\alpha\big(g(s)\big)\,\mathrm{d}s, \quad t \ge t_0.$$

By Lemma 2.2 there exist a $T \ge t_0$, $\lambda \in (0,1)$ and $k \in \{1,3,\dots,2n-1\}$ such that

$$x(t) \ge H_0\big(t,T;a;k;\lambda\big)L_{2n-1}^{1/\alpha}x(t) \quad \text{for} \quad t \ge T. \tag{3.33}$$

Thus we have

$$\begin{aligned} x^\alpha(t) &\ge H_0^\alpha(t,T;a;k;\lambda)L_{2n-1}x(t) \\ &\ge H_0^\alpha(t,T;a;k;\lambda) \int_t^\infty q(s)x^\alpha\big(g(s)\big)\,\mathrm{d}s, \quad t \ge T. \end{aligned}$$

Now by $\mu(t) \ge t$ and the fact that $x'(t) > 0$ and $g(s) \ge t$ for $s \ge \mu(t)$ it follows that

$$\begin{aligned} x^\alpha(t) &\ge H_0^\alpha(t,T;a;k;\lambda)\Bigg(\int_{\mu(t)}^\infty q(s)x^\alpha\big(g(s)\big)\,\mathrm{d}s\Bigg) \\ &\ge H_0^\alpha(t,T;a;k;\lambda)x^\alpha(t)\Bigg(\int_{\mu(t)}^\infty q(s)\,\mathrm{d}s\Bigg). \end{aligned} \tag{3.34}$$

Dividing both sides of (3.34) by $x^\alpha(t)$ we have

$$H_0^\alpha(t,T;a;k;\lambda)\int_{\mu(t)}^{\infty} q(s)\,\mathrm{d}s \le 1, \quad t\ge T. \tag{3.35}$$

Taking lim sup of both sides of (3.35) as $t\to\infty$ we obtain a contradiction to condition (3.32). This completes the proof. □

In the case of an *advanced* equation (1.1), i.e., $g(t)\ge t$ for $t\ge t_0$, Theorem 3.7 takes the following form.

Theorem 3.8. *Let $g(t)\ge t$ for $t\ge t_0$ and conditions* (1.2) *and* (3.10) *hold with $\alpha=\beta$. If for all large $T\ge t_0, k\in\{1,3,\dots,2n-1\}$ and some constant λ, $\lambda\in(0,1)$,*

$$\limsup_{t\to\infty} H_0^\alpha(t,T;a;k;\lambda)\int_t^\infty q(s)\,\mathrm{d}s > 1, \tag{3.36}$$

then equation (1.1) *is oscillatory.*

Next we present the following result when

$$Q(t) := \int_t^\infty q(s)\,\mathrm{d}s < \infty \quad \text{for} \quad t\ge t_0. \tag{3.37}$$

Theorem 3.9. *Let conditions* (1.2), (3.2) *with $\sigma'(t)\ge 0$ for $t\ge t_0$,* (3.10) *with $\alpha=\beta$ and* (3.37) *hold. If for $k\in\{1,3,\dots,2n-1\}$, some constant $\lambda,\lambda\in(0,1)$ and all large $T\ge t_0$ with $\sigma(t)>T/\lambda$,*

$$\limsup_{t\to\infty} H_0\big(\sigma(t),T;a;k;\lambda\big)\Big(Q(t) + \alpha\lambda\int_t^\infty H_1\big(\sigma(t),T;a;k;\lambda\big)\sigma'(s)Q^{(\alpha+1)/\alpha}(s)\,\mathrm{d}s\Big)^{1/\alpha} > 1, \tag{3.38}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t)>0$ for $t\ge t_0\ge 0$. Define $w(t)$ as in (3.5) with $\rho(t)=1$ and $f(x)=x^\alpha$ and as in the proof of Theorem 3.1 we obtain (3.7) which takes the form

$$w'(t)\le -q(t)-\alpha\lambda\sigma'(t)H_1\big(\sigma(t),T;a;k;\lambda\big)w^{1+1/\alpha}(t), \quad t\ge T\ge t_0. \tag{3.39}$$

Integrating (3.39) from t $(\ge T)$ to $u(\ge t)$ and letting $u\to\infty$ we find that

$$\frac{L_{2n-1}^{1/\alpha}x(t)}{x\big(\lambda\sigma(t)\big)} \ge \Big(Q(t)+\alpha\lambda\int_t^\infty H_1\big(\sigma(s),T;a;k;\lambda\big)w^{1+1/\alpha}(s)\,\mathrm{d}s\Big)^{1/\alpha}, \quad t\ge T. \tag{3.40}$$

Now one can easily see that

$$w(t)\ge Q(t) \quad \text{for} \quad t\ge T. \tag{3.41}$$

Using (3.33) with $t = \sigma(t)$ and (3.41) in (3.40) we have

$$1 \geq H_0\big(\sigma(t), T; a; k; \lambda\big)\left(Q(t) + \alpha\lambda\int_t^\infty H_1\big(\sigma(s), T; a; k; \lambda\big) Q^{1+1/\alpha}(s)\,\mathrm{d}s\right)^{1/\alpha}.$$

Taking lim sup of both sides of the above inequality as $t \to \infty$ we obtain a contradiction to condition (3.38). This completes the proof. □

Next we have the following comparison result.

Theorem 3.10. *Let conditions* (1.2), (3.2) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$ *and* (3.10) *hold. If for* $k \in \{1, 3, \dots, 2n-1\}$, *some constant* λ, $\lambda \in (0,1)$, *and all large* $T \geq t_0$ *with* $\sigma(t) > T/\lambda$, *every solution of the first-order delay differential equation*

$$y'(t) + q(t)H_0^\beta\big(\sigma(t), T; a; k; \lambda\big)y^{\beta/\alpha}\big(\sigma(t)\big) = 0 \tag{3.42}$$

is oscillatory, then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 3.7 we obtain (3.33) for $t \geq T$. There exists a $T_0 \geq T$ such that

$$x\big(\sigma(t)\big) \geq H_0\big(\sigma(t), T; a; k; \lambda\big)\, L_{2n-1}^{1/\alpha} x\big(\sigma(t)\big), \quad t \geq T_0. \tag{3.43}$$

Using condition (3.10) and (3.43) in equation (1.1) we have

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t} L_{2n-1}x(t) &= q(t)f\big(x\big(g(t)\big)\big) \geq q(t)x^\beta\big(\sigma(t)\big) \\ &\geq q(t)H_0^\beta\big(\sigma(t), T; a; k; \lambda\big)L_{2n-1}^{\beta/\alpha}x\big(\sigma(t)\big), \quad t \geq T_0. \end{aligned}$$

Set $y(t) = L_{2n-1}x(t) > 0,\ t \geq T_0$. We get

$$y'(t) + q(t)H_0^\beta\big(\sigma(t), T; a; k; \lambda\big)y^{\beta/\alpha}\big(\sigma(t)\big) \leq 0, \quad t \geq T_0. \tag{3.44}$$

Integrating the inequality (3.44) from $t(\geq T_0)$ to u and letting $u \to \infty$ we have

$$y(t) \geq \int_t^\infty q(s)H_0^\beta\big(\sigma(t), T; a; k; \lambda\big)\, y^{\beta/\alpha}\big(\sigma(s)\big)\,\mathrm{d}s, \quad t \geq T_0.$$

As in [**17**] it is easy to conclude that there exists a positive solution $y(t)$ of equation (3.42) with $\lim_{t\to\infty} y(t) = 0$, which contradicts the fact that equation (3.42) is oscillatory. This completes the proof. □

The following corollary is immediate.

Corollary 3.1. *Let conditions* (1.2), (3.2) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$ *and* (3.10) *hold. If for* $k \in \{1, 3, \dots, 2n-1\}$, *some constant* λ, $\lambda \in (0,1)$, *and all large* $T \geq t_0$ *with* $\sigma(t) > T/\lambda$, *either*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^t q(s)H_0^\alpha\big(\sigma(s), T; a; k; \lambda\big)\,\mathrm{d}s > \frac{1}{e}, \quad \textit{when} \ \ \alpha = \beta, \tag{3.45}$$

or

$$\lim_{t\to\infty} \int^{t} q(s) H_0^{\beta}\big(\sigma(s), T; a; k; \lambda\big)\, \mathrm{d}s = \infty, \quad \textit{when } \beta < \alpha \tag{3.46}$$

holds, then equation (1.1) *is oscillatory.*

REMARK 3.1. We note that some of our results of this section are new even when $\alpha = 1$.

4. SUFFICIENT CONDITIONS

In this section we present some criteria for the oscillation of equation (1.1) when the function f satisfies either

$$\int^{\pm\infty} \frac{\mathrm{d}u}{f^{1/\alpha}(u)} < \infty \tag{4.1}$$

or

$$\int_{\pm 0} \frac{\mathrm{d}u}{f(u^{1/\alpha})} < \infty. \tag{4.2}$$

Theorem 4.1. *Let* $\alpha \geq 1$ *and conditions* (1.2), (3.2) *and* (4.1) *hold. Moreover assume that there exist a function* $\rho \in C^1\big([t_0, \infty), \mathbb{R}^+\big)$, *a constant* λ, $\lambda \in (0, 1)$, *and* $k \in \{1, 3, \ldots, 2n-1\}$ *such that for all large* $T \geq t_0$ *with* $\sigma(t)T/\lambda$

$$\rho'(t) \geq 0 \quad \textit{and} \quad \left(\frac{(\rho'(t))^{1/\alpha}}{H_1\big(\sigma(t), T; a; k; \lambda\big)}\right)' \leq 0, \quad t \geq T. \tag{4.3}$$

If

$$\int^{\infty} \rho(s) q(s)\, \mathrm{d}s = \infty, \tag{4.4}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 3.1 we define the function $w(t)$ as in (3.5) and proceed to obtain (3.4) and (3.6), i.e.,

$$w'(t) \leq -\rho(t) q(t) + \rho'(t)\, \frac{L_{2n-1} x(t)}{f\big(x\big(\lambda\sigma(t)\big)\big)}, \quad t \geq T \geq t_0. \tag{4.5}$$

Using (3.4) in (4.5) we get

$$w'(t) \leq -\rho(t) q(t) + \frac{\rho'(t)}{f\big(x\big(\lambda\sigma(t)\big)\big)} \left(\frac{x'\big(\lambda\sigma(t)\big)\lambda\sigma'(t)}{H_1\big(\sigma(t), T; a; k; \lambda\big)\lambda\sigma'(t)}\right)^{\alpha} \tag{4.6}$$

$$= -\rho(t)q(t) + \left(\left(\frac{\big(\rho'(t)\big)^{1/\alpha}}{H_1\big(\sigma(),T;a;k;\lambda\big)\lambda\sigma'(t)}\right)\frac{x'\big(\lambda\sigma(t)\big)\lambda\sigma'(t)}{f^{1/\alpha}\big(x\big(\lambda\sigma(t)\big)\big)}\right)^{\alpha},\ t \geq T_1 \geq T.$$

Integrating (4.6) from T_1 to t we obtain

$$\begin{aligned} w(t) &\leq w(T_1) - \int_{T_1}^{t} \rho(s)q(s)\,\mathrm{d}s \\ &\quad + \int_{T_1}^{t} \left(\left(\frac{\big(\rho'(s)\big)^{1/\alpha}}{H_1\big(\sigma(s),T;a;k;\lambda\big)\lambda\sigma'(s)}\right)\frac{x'\big(\lambda\sigma(s)\big)\lambda\sigma'(s)}{f^{1/\alpha}\big(x\big(\lambda\sigma(s)\big)\big)}\right)^{\alpha}\mathrm{d}s \\ &\leq w(T_1) - \int_{T_1}^{t} \rho(s)q(s)\,\mathrm{d}s \\ &\quad + \left(\int_{T_1}^{t} \left(\frac{\big(\rho'(s)\big)^{1/\alpha}}{H_1\big(\sigma(s),T;a;k;\lambda\big)\lambda\sigma'(s)}\right)\frac{x'\big(\lambda\sigma(s)\big)\lambda\sigma'(s)}{f^{1/\alpha}\big(x\big(\lambda\sigma(s)\big)\big)}\,\mathrm{d}s\right)^{\alpha}. \end{aligned} \tag{4.7}$$

However, by the BONNET second mean-value theorem, for a fixed $t \geq T_1$ and for some $\xi \in [T_1, t]$, we have

$$\begin{aligned} &\int_{T_1}^{t} \left(\frac{\big(\rho'(s)\big)^{1/\alpha}}{H_1\big(\sigma(s),T;a,k;\lambda\big)\lambda\sigma'(s)}\right)\left(\frac{x'\big(\lambda\sigma(s)\big)\lambda\sigma'(s)}{f^{1/\alpha}\big(x\big(\lambda\sigma(s)\big)\big)}\right)\mathrm{d}s \\ &\qquad = \left(\frac{\big(\rho'(T_1)\big)^{1/\alpha}}{H_1\big(\sigma(T_1),T;a;k;\lambda\big)\lambda\sigma'(T_1)}\right)\int_{x(\lambda\sigma(T_1))}^{x(\lambda\sigma(t))} \frac{\mathrm{d}u}{f^{1/\alpha}(u)} \\ &\qquad \leq \left(\frac{\big(\rho'(T_1)\big)^{1/\alpha}}{H_1\big(\sigma(T_1),T;a;k;\lambda\big)\lambda\sigma'(T_1)}\right)\int_{x(\lambda\sigma(T_1))}^{\infty} \frac{\mathrm{d}u}{f^{1/\alpha}(u)} := M, \end{aligned} \tag{4.8}$$

where M is a positive constant.

Using (4.8) in (4.7) we have

$$\int_{T_1}^{t} \rho(s)q(s)\,\mathrm{d}s \leq -w(t) + w(T_1) + M^{\alpha}. \tag{4.9}$$

Letting $t \to \infty$ in (4.9), we arrive at a contradiction to condition (4.4) and this completes the proof. □

The following result is immediate.

Theorem 4.2. *Let condition* (4.3) *in Theorem* 4.1 *be replaced by*

$$\rho'(t) \geq 0 \ \ \textit{for} \ \ t \geq t_0 \ \ \textit{and} \ \ \int^{\infty} \left|\left(\frac{\big(\rho'(s)\big)^{1/\alpha}}{\sigma'(s)H_1\big(\sigma(s),T;a;k;\lambda\big)}\right)'\right|\mathrm{d}s < \infty. \tag{4.10}$$

Then the conclusion of Theorem 4.1 *holds.*

Next we present the following oscillation criteria for equation (1.1) when condition (3.37) is satisfied.

Theorem 4.3. *Let conditions* (1.2), (3.2) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$, (3.37) *and* (4.1) *hold. If for all large* $T \geq t_0$, *some constant* λ, $\lambda \in (0,1)$ *and* $k \in \{1,3,\dots,2n-1\}$ *such that for* $\sigma(t) > T/\lambda$,

$$\int^{\infty} H_1\big(\sigma(s),T;a;k;\lambda\big)\sigma'(s)Q^{1/\alpha}(s)\,\mathrm{d}s = \infty, \tag{4.11}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Define the function $w(t)$ as in (3.5) with $\rho(t) = 1$. Then we obtain

$$\int_{t_1}^{t} q(s)\,\mathrm{d}s \leq \frac{L_{2n-1}x(t_1)}{f\big(x\big(\lambda\sigma(t_1)\big)\big)}$$

and hence for any $t \geq t_1$

$$Q^{1/\alpha}(t) \leq \frac{L_{2n-1}^{1/\alpha}x(t)}{f^{1/\alpha}\big(x\big(\lambda\sigma(t)\big)\big)}. \tag{4.12}$$

Using (3.4) in (4.12) we obtain

$$H_1\big(\sigma(t),T;a;k;\lambda\big)\lambda\sigma'(t)Q^{1/\alpha}(t) \leq \frac{x'\big(\lambda\sigma(t)\big)\lambda\sigma'(t)}{f^{1/\alpha}\big(x\big(\lambda\sigma(t)\big)\big)} \tag{4.13}$$

for $\sigma(t) > T/\lambda$, $T \geq t_1$.

Integrating (4.13) from T to t we get

$$\begin{aligned}\lambda\int_T^t H_1\big(\sigma(s),T;a;k;\lambda\big)\sigma'(s)Q^{1/\alpha}(s)\,\mathrm{d}s &\leq \int_{x(\lambda\sigma(T))}^{x(\lambda\sigma(t))} \frac{\mathrm{d}u}{f^{1/\alpha}(u)}\\ &\leq \int_{x(\lambda\sigma(T))}^{\infty} \frac{\mathrm{d}u}{f^{1/\alpha}(u)} < \infty,\end{aligned}$$

which contradicts condition (4.11) and completes the proof. □

Theorem 4.4. *Let conditions* (1.2), (3.1), (3.2) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$, (3.37) *and* (4.1) *hold. If for all large* $T \geq t_0$ *with* $\sigma(t) > T/\lambda$ *for some constant* λ, $\lambda \in (0,1)$, *and* $k \in \{1,3,\dots,2n-1\}$,

$$\int^{\infty} H_1\big(\sigma(s),T;a;k;\lambda\big)\sigma'(s)\bigg(Q(s) + \overline{k}\lambda\int_s^{\infty} H_1\big(\sigma(u),T;a;k;\lambda\big)\sigma'(u)Q^{1+1/\alpha}(u)du\bigg)^{1/\alpha} \mathrm{d}s = \infty, \tag{4.14}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Define the function $w(t)$ as in (3.5) with $\rho(t) = 1$. Then we obtain

$$w'(t) \leq -q(t) - \frac{L_{2n-1}x(t)}{f^2\big(x\big(\lambda\sigma(t)\big)\big)}\,\lambda\sigma'(t)x'\big(\lambda\sigma(t)\big), \quad t \geq t_1 \geq t_0. \tag{4.15}$$

Using (3.4) and (3.1) in (4.15) we get

$$w'(t) \leq -q(t) - \lambda\overline{k}\sigma'(t)H_1\big(\sigma(t),T;a;k;\lambda)w^{1+1/\alpha}\big)(t), \quad t \geq T \geq t_1. \tag{4.16}$$

Integrating (4.16) from $t(\geq T)$ to $u(\geq t)$ and letting $u \to \infty$ we obtain

$$L_{2n-1}x(t) \geq f\big(x\big(\lambda\sigma(t)\big)\big)\bigg(Q(t) + \lambda\overline{k}\int_t^{\infty} H_1\big(\sigma(s),T;a;k;\lambda\big)\sigma'(s)w^{1+1/\alpha}(s)\,\mathrm{d}s\bigg), \quad t \geq T, \tag{4.17}$$

and

$$w(t) \geq Q(t), \quad t \geq T. \tag{4.18}$$

Using (3.4) and (4.18) in (4.17) we find

$$\frac{x'\big(\lambda\sigma(t)\big)\lambda\sigma'(t)}{f^{1/\alpha}\big(x\big(\lambda\sigma(t)\big)\big)} \geq \lambda\sigma'(t)H_1\big(\sigma(t),T;a;k;\lambda\big)\bigg(Q(t) + \lambda\overline{k}\int_t^{\infty} H_1\big(\sigma(s),T;a;k;\lambda\big)\sigma'(s)Q^{1+1/\alpha}(s)ds\bigg)^{1/\alpha}.$$

Integrating the above inequality from T to t and using condition (4.1) we obtain a contradiction to condition (4.14) and complete the proof. □

Next we present the following theorem when condition (4.2) holds.

Theorem 4.5. *Let conditions* (1.2), (3.2) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$ *and* (4.2) *hold. Moreover assume that*

$$-f(-xy) \geq f(xy) \geq f(x)f(y) \quad \textit{for} \quad xy > 0. \tag{4.19}$$

If for all large $T \geq t_0$ *with* $\sigma(t) > T/\lambda$ *for some constant* λ, $\lambda \in (0,1)$ *and* $k \in \{1, 3, \ldots, 2n-1\}$,

$$\int^{\infty} q(s)f\big(H_0\big(\sigma(s),T;a;k;\lambda\big)\big)\,\mathrm{d}s = \infty, \tag{4.20}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 3.10 there exists a $T_0 \geq T$ such that (3.43) holds for $t \geq T_0$.

Using (3.43) and (4.19) in equation (1.1) we get

$$\begin{aligned} -L_{2n}x(t) &= q(t)f\big(x\big(g(t)\big)\big) \geq q(t)f\big(x\big(\sigma(t)\big)\big) \\ &\geq q(t)f\big(H_0\big(\sigma(t),T;a;k;\lambda\big)L_{2n-1}^{1/\alpha}x(t)\big) \\ &\geq q(t)f\big(H_0\big(\sigma(t),T;a;k;\lambda\big)\big)f\big(L_{2n-1}^{1/\alpha}x(t)\big), \quad t \geq T_0. \end{aligned} \tag{4.21}$$

Let $u(t) = L_{2n-1}x(t)$ for $t \geq T_0$. We have

$$-\frac{\mathrm{d}u(t)}{\mathrm{d}t} \geq q(t)f\big(H_0\big(\sigma(t),T;a;k;\lambda\big)\big)f\big(u^{1/\alpha}(t)\big), \quad t \geq T_0. \tag{4.22}$$

Dividing both sides of (4.22) by $f\big(u^{1/\alpha}(t)\big)$ and integrating from T_0 to t we have

$$\int_{T_0}^{t} q(s)f\big(H_0\big(\sigma(s),T;a;k;\lambda\big)\big)\,\mathrm{d}s \leq \int_{t}^{T_0} \frac{u'(s)\,\mathrm{d}s}{f\big(u^{1/\alpha}(s)\big)} = \int_{u(t)}^{u(T_0)} \frac{\mathrm{d}u}{f\big(u^{1/\alpha}\big)}\,.$$

Letting $t \to \infty$ we conclude that

$$\int_{T_0}^{\infty} q(s)f\big(H_0\big(\sigma(s),T;a;k;\lambda\big)\big)\,\mathrm{d}s \leq \int_{0}^{u(T_0)} \frac{\mathrm{d}u}{f(u^{1/\alpha})} < \infty\,,$$

which contradicts condition (4.20). This completes the proof. □

Theorem 4.6. *Let conditions* (1.2), (3.2) *with* $\sigma'(t) \geq 0$ *for* $t \geq t_0$, (3.10) *with* $\beta < \alpha$ *and* (3.37) *hold. If for all constant* $c > 0$, $T \geq t_0$ *with* $\sigma(t) > T/\lambda$ *for some constant* λ, $\lambda \in (0,1)$ *and* $k \in \{1,3,\dots,2n-1\}$,

$$\limsup_{t\to\infty} Q^{1/\beta}(t)H_0\big(\sigma(t),T;a;k;\lambda\big)\Bigg(1 + \frac{c}{Q(t)}\int_t^\infty H_1\big(\sigma(s),T;a;k;\lambda\big)\sigma'(s)Q^{1+1/\beta}(s)\,\mathrm{d}s\Bigg)^{1/\alpha} = \infty, \tag{4.23}$$

then equation (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Define $w(t) = L_{2n-1}x(t)/x^{\beta}\big(\lambda\sigma(t)\big)$ for $t \geq t_1 \geq t_0$. Then for $t \geq t_1$ we have

$$w'(t) \leq -q(t) - \lambda\beta\sigma'(t)\,\frac{L_{2n-1}x(t)}{x^{\beta+1}\big(\lambda\sigma(t)\big)}x'\big(\lambda\sigma(t)\big).$$

As in the proof of Theorems 3.1 and 3.10 we obtain (3.4) and (3.43), respectively, for $t \geq T_0 \geq T \geq t_1$,

$$w'(t) \leq -q(t) - \lambda\beta\sigma'(t)w^{1+1/\alpha}(t)x^{\beta/\alpha-1}\big(\lambda\sigma(t)\big), \quad t \geq T_0. \tag{4.24}$$

Integrating (4.24) from t ($\geq T_0$) to u and letting $u \to \infty$ we find

$$L_{2n-1}x(t) \geq x^{\beta}\big(\lambda\sigma(t)\big)\Big(Q(t) + \lambda\beta\int_t^{\infty}\sigma'(s)H_1\big(\sigma(s),T;a;k;\lambda\big)w^{1+1/\alpha}(s)x^{\beta/\alpha-1}\big(\lambda\sigma(s)\big)\,\mathrm{d}s\Big), \quad t \geq T_0, \tag{4.25}$$

and

$$w(t) \geq Q(t), \quad t \geq T_0.$$

There exist a constant $c_1 > 0$ and a $T_1 \geq T_0$ such that

$$L_{2n-1}x(t) \leq c_1, \quad t \geq T_1. \tag{4.26}$$

Now for $t \geq T_1$ it follows from (4.25) and (4.26) that

$$x^{\beta/\alpha}\big(\lambda\sigma(t)\big) \leq c_1 Q^{1/\alpha}(t) \quad \text{or} \quad x\big(\lambda\sigma(t)\big) \leq c_1^{\alpha/\beta}Q^{-1/\beta}(t)$$

and hence

$$x^{\beta/\alpha-1}\big(\lambda\sigma(t)\big) \geq c_1^{1-\alpha/\beta}Q^{1/\beta-1/\alpha}(t), \quad t \geq T_1. \tag{4.27}$$

Using (4.27) in (4.25) yields

$$L_{2n-1}^{1/\alpha}x(t) \geq x^{\beta/\alpha}\big(\lambda\sigma(t)\big)\Big(Q(t) + \lambda\beta c_1^{1-\alpha/\beta}\int_t^{\infty}H_1\big(\sigma(s),T;a;k;\lambda\big)\sigma'(s)Q^{1+1/\beta}(s)\,\mathrm{d}s\Big)^{1/\alpha}. \tag{4.28}$$

Using (3.43) in (4.28) we obtain for $T \geq T_1$

$$\begin{aligned} x\big(\lambda\sigma(t)\big) &\geq H_0\big(\sigma(t),T;a;k;\lambda\big)L_{2n-1}^{1/\alpha}x(t) \\ &\geq x^{\beta/\alpha}\big(\lambda\sigma(t)\big)H_0\big(\sigma(t),T;a;k;\lambda\big)\Big(Q(t) \\ &\qquad + \lambda\beta c_1^{\beta/\alpha-1}\int_t^{\infty}H_1\big(\sigma(s),T;a;k,\lambda\big)\sigma'(s)Q^{1+1/\beta}(s)\,\mathrm{d}s\Big)^{1/\alpha} \end{aligned}$$

or

$$x^{1-\beta/\alpha}\big(\lambda\sigma(t)\big) \geq H_0\big(\sigma(t),T;a;k;\lambda\big)Q^{1/\alpha}(t)\Big(1 + \frac{\lambda\beta c_1^{1-\alpha/\beta}}{Q(t)}\int_t^{\infty}H_1\big(\sigma(s),T;a;k,\lambda\big)\sigma'(s)Q^{1+1/\beta}(s)\,\mathrm{d}s\Big)^{1/\alpha}.$$

Using (4.27) in the above inequality one can easily see that

$$c_1^{\alpha/\beta-1} \ge Q^{1/\beta}(t)H_0\big(\sigma(t),T;a;k;\lambda\big)\Bigg(1 + \frac{\lambda\beta c_1^{1-\alpha/\beta}}{Q(t)}\int_t^\infty H_1\big(\sigma(s),T;a;k,\lambda\big)\sigma'(s)Q^{1+1/\beta}(s)\,\mathrm{d}s\Bigg)^{1/\alpha},\ t \ge T_1.$$

Taking lim sup of both sides of this inequality as $t \to \infty$ we obtain a contradiction to condition (4.23). This completes the proof. □

5. NECESSARY AND SUFFICIENT CONDITIONS

In this section we are interested to establish some necessary and sufficient conditions for the oscillation of equation (1.1). Here for $t \ge T \ge t_0$ we let

$$H_*(t,T;a) = \int_T^t \frac{(t-u)^{n-1}}{(n-1)!}\bigg(\frac{(u-T)^{n-1}}{(n-1)!\,a(u)}\bigg)^{1/\alpha}\mathrm{d}u.$$

Theorem 5.1. *Let condition* (1.2) *hold,* $f(x)\operatorname{sgn} x = |x|^\beta$ *for* $x \ne 0$ *and* $\beta < \alpha, g(t) \le t$ *and* $g'(t) \ge 0$ *for* $t \ge t_0$. *Equation* (1.1) *is oscillatory if and only if for all large* $T \ge t_0$

$$\int^\infty q(s)H_*^\beta\big(g(s),T;a\big)\,\mathrm{d}s = \infty. \tag{5.1}$$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \ge t_0 \ge 0$. The proof of the "if" part is similar to that of Theorem 4.5 and we omit the details. To prove the "only if" part it suffices to assume that for all large $\overline{T} \ge t_0$

$$\int^\infty q(s)H_*^\beta\big(g(s),\overline{T};a\big)\,\mathrm{d}s < \infty \tag{5.2}$$

and to show the existence of a nonoscillatory solution of equation (1.1). Here we give an outline of the proof.

Let $c > 0$ be an arbitrary constant and choose $T \ge \overline{T}$ sufficiently large so that

$$\int_T^\infty q(s)H_*^\beta\big(g(s),\overline{T};a\big)\,\mathrm{d}s \le 2^{-1/2}c^{1-\beta/\alpha}. \tag{5.3}$$

Define the set X by

$$X = \big\{x \in C[T,\infty) : c_1H_*(t,T;a) \le x(t) \le c_2H_*(t,T;a),\quad t \ge T\big\} \tag{5.4}$$

which is a closed convex subset of the locally convex space $C[T,\infty)$ of continuous functions on $[T,\infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T,\infty)$, where c_1 and c_2 denote the positive constants

$$c_1 = c^{1/\alpha} \quad \text{and} \quad c_2 = (2c)^{1/\alpha}. \tag{5.5}$$

Consider the integral operator $\mathcal{T}$ defined by

$$\mathcal{T}x(t) = \int_T^t \frac{(t-s)^{n-1}}{(n-1)!}\Bigg(\frac{1}{a(s)}\bigg(c\,\frac{(s-T)^{n-1}}{(n-1)!} + \int_T^s \frac{(s-u)^{n-2}}{(n-2)!}\int_u^\infty q(\tau)x^\beta\big(g(\tau)\big)\,\mathrm{d}\tau\mathrm{d}u\bigg)^{1/\alpha}\Bigg)\mathrm{d}s, \quad t > T. \tag{5.6}$$

Using (5.3) and (5.5) we see that $\mathcal{T}$ maps X into itself. If $\{x_j\}$ is a sequence in X converging to x_0 in $C[T,\infty)$, then from the Lebesgue Monotone Convergence Theorem it follows that $\{\mathcal{T}x_j\}$ converges to $\mathcal{T}x_0$ in $C[T,\infty)$ so that $\mathcal{T}$ is a continuous mapping. Since $\mathcal{T}(X)$ and $\mathcal{T}'(X) = \{(\mathcal{T}x)'(t) : x \in X\}$ are locally bounded in $[T,\infty)$, the Ascoli–Arzela Theorem implies that $\mathcal{T}(X)$ is relatively compact in $C[T,\infty)$. Thus all the hypotheses of Schauder–Tychonov fixed point theorem are satisfied and so there exists an element $x \in X$ such that $x = \mathcal{T}x$. Differentiating the integral equation $x = \mathcal{T}x$ we conclude that $x = x(t)$ is a positive solution of equation (1.1) on $[T,\infty)$ such that $\lim_{t\to\infty} x(t)/H_*(t,T;a) = c$. This completes the proof. □

Before we prove the next result we state the following theorem.

Theorem 5.2. *Let condition* (1.2) *hold. If*

$$\int^\infty s^{n-1}\bigg(\frac{1}{a(s)}\int_s^\infty u^{n-1}q(u)\,\mathrm{d}u\bigg)^{1/\alpha}\mathrm{d}s = \infty, \tag{5.7}$$

then equation (1.1) *is oscillatory.*

Proof. The proof is immediate. □

Theorem 5.3. *Let condition* (1.2) *hold,* $f(x)\operatorname{sgn} x = |x|^\beta$ *for* $x \neq 0$ *and* $\beta > \alpha$, $g(t) \le t$ *and* $g'(t) \ge 0$ *for* $t \ge t_0$. *Equation* (1.1) *is oscillatory if and only if* (5.7) *holds.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \ge t_0 \ge 0$. The proof of the "if" part is the same as that of Theorem 5.2 and hence is omitted. The "only if" part is proved as follows: Let $c > 0$ be given arbitrarily and choose $T \ge t_0$ so that

$$\int_T^\infty \frac{t^{n-1}}{(n-1)!}\bigg(\frac{1}{a(t)}\int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!}\,q(s)\,\mathrm{d}s\bigg)^{1/\alpha}\mathrm{d}t < \frac{1}{2}\,c^{1-\beta/\alpha}.$$

We define the set Y and the mapping $\mathcal{S}$ by

$$Y = \Big\{x \in C[T,\infty) : \frac{c}{2} \le x(t) \le c,\ t \ge T\Big\}$$

and

$$\mathcal{S}x(t) = c - \int_t^\infty \frac{(s-t)^{n-1}}{(n-1)!} \left(\frac{1}{a(s)} \int_s^\infty \frac{(u-s)^{n-1}}{(n-1)!} q(u) x^\beta\big(g(u)\big)\,\mathrm{d}u \right)^{1/\alpha} ds, \quad t \geq T,$$

respectively. Then it is easy to show that $\mathcal{S}$ maps Y into itself, that $\mathcal{S}$ is a continuous mapping and $\mathcal{S}(Y)$ is relatively compact in $C[T,\infty)$. Therefore by the SCHAUDER–TYCHONOV fixed point theorem there exists an element $x \in Y$ such that $x = \mathcal{S}x$. It is clear that the fixed point $x = x(t)$ gives a positive solution of equation (1.1) on $[T,\infty)$ such that $\lim_{t\to\infty} x(t) = c$. This completes the proof. □

6. MORE COMPARISON RESULTS

In this section we compare the inequality

$$L_{2n}x(t) + q(t) f\big(x\big(g(t)\big)\big) \leq 0 \quad (\geq 0) \tag{6.1}$$

with equation (1.1). In fact we establish the following theorem.

Theorem 6.1. *Let condition* (1.2) *hold. If inequality* (4.1) *has an eventually positive* (*negative*) *solution, then equation* (1.1) *also has an eventually positive* (*negative*) *solution.*

Proof. Let $x(t)$ be an eventually positive solution of inequality (6.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. According to Lemma 2.1 there exist a $t_1 \geq t_0$ and an integer $k \in \{1, 3, \ldots, 2n-1\}$ such that inequalities (2.1) hold. Here we distinguish the following three cases: (I) $k = 2n-1$, (II) $n+1 \leq k \leq 2n-3$, (III) $1 \leq k \leq n$. For this, when we integrate inequality (6.1) from t to $u(\geq t \geq t_1)$ and let $u \to \infty$, we have

$$L_{2n-1}x(t) \geq \int_t^\infty q(s) f\big(x\big(g(s)\big)\big)\,\mathrm{d}s. \tag{6.2}$$

Case (I) Let $k = 2n-1$. Integrating (6.2) $(n-1)$ times from t_1 to t we obtain

$$\begin{aligned} x^{(n)}(t) &\geq \left(\frac{1}{a(t)} \int_{t_1}^t \int_{t_1}^{s_{n+1}} \cdots \int_{t_1}^{s_{2n-2}} \int_{s_{2n-1}}^\infty q(s) f\big(x\big(g(s)\big)\big)\,\mathrm{d}s\mathrm{d}s_{2n-1}\cdots\mathrm{d}s_{n+1} \right)^{1/\alpha} \\ &:= \Phi_1\big(t; x\big(g(t)\big)\big) \quad \text{for} \quad t \geq t_1 \end{aligned} \tag{6.3}$$

from which after integrating n times from t_1 to t it follows that

$$\begin{aligned} x(t) &\geq x(t_1) + \int_{t_1}^t \int_{t_1}^{s_1} \cdots \int_{t_1}^{s_{n-1}} \Phi_1(s_n, x)\,\mathrm{d}s_n\mathrm{d}s_{n-1}\cdots\mathrm{d}s_1 \\ &:= x(t_1) + \Psi_1\big(t; x\big(g(t)\big)\big) \quad \text{for} \quad t \geq t_1. \end{aligned} \tag{6.4}$$

Case (II) Let $n+1 \le k \le 2n-3$. Integrating (6.2) $(2n-k-1)$ times from t to $u(\ge t)$ and letting $u \to \infty$ yield

$$(-1)^{2n-k-1} L_k x(t) \ge \int_t^\infty \int_{s_{2n-k-1}}^\infty \cdots \int_{s_{2n-1}}^\infty q(s) f\big(x\big(g(s)\big)\big)\,\mathrm{d}s\mathrm{d}ds_{2n-1}\cdots \mathrm{d}s_{2n-k-1}.$$

Integrating this inequality $(k-n)$ times from t_1 to t we have

$$\text{(6.5)}\qquad \begin{aligned} x^{(n)}(t) \ge \Bigg(\frac{1}{a(t)}\int_{t_1}^{t}\int_{t_1}^{s_{n+1}}\cdots\int_{t_1}^{s_{2n-k-3}}\int_{s_{2n-k-2}}^{\infty} \\ \cdots\int_{s_{2n-1}}^{\infty} q(s) f\big(x\big(g(s)\big)\big)\,\mathrm{d}s\mathrm{d}s_{2n-1}\cdots\mathrm{d}s_{n+1}\Bigg)^{1/\alpha} \\ := \Phi_2\big(t; x\big(g(t)\big)\big) \quad \text{for} \quad t \ge t_1. \end{aligned}$$

Integrating (6.5) n times from t_1 to t we get

$$\text{(6.6)}\qquad \begin{aligned} x(t) \ge x(t_1) + \int_{t_1}^{t}\int_{t_1}^{s_1}\cdots\int_{t_1}^{s_{n-1}} \Phi_2\big(s_n; x\big(g(s_n)\big)\big)\,\mathrm{d}s_n\mathrm{d}s_{n-1}\cdots\mathrm{d}s_1 \\ := x(t_1) + \Psi_2\big(t; x\big(g(t)\big)\big) \quad \text{for} \quad t \ge t_1. \end{aligned}$$

Case (III) Let $1 \le k \le n$. Integrating (6.2) $(n-1)$ times from t to $u(\ge t)$ and letting $u \to \infty$ we have

$$\text{(6.7)}\qquad \begin{aligned} (-1)^n x^{(n)}(t) \ge \Bigg(\frac{1}{a(t)}\int_{t}^{\infty}\int_{s_{n+1}}^{\infty} \\ \cdots\int_{s_{2n-1}}^{\infty} q(s) f\big(x\big(g(s)\big)\big)\,\mathrm{d}s\mathrm{d}s_{2n-1}\cdots\mathrm{d}s_{n+1}\Bigg)^{1/\alpha} \\ := \Phi_3\big(t; x\big(g(t)\big)\big) \quad \text{for} \quad t \ge t_1. \end{aligned}$$

Integrating (6.7) $(n-k)$ times from t to $u(\ge t)$ and letting $u \to \infty$ we have

$$(-1)^{2n-k-1} L_k x(t) \ge \int_t^\infty \int_{s_{k-1}}^\infty \cdots \int_{s_{n-1}}^\infty \Phi_3\big(s_n; x\big(g(s_n)\big)\big)\,\mathrm{d}s_n\mathrm{d}s_{n-1}\cdots \mathrm{d}s_{k-1}.$$

Further repeated integration of the above inequality shows that

$$\text{(6.8)}\qquad \begin{aligned} x(t) \ge x(t_1) + \int_{t_1}^{t}\int_{t_1}^{s_1}\cdots\int_{t_1}^{s_{k-1}}\int_{s_k}^{\infty}\cdots\int_{s_{n-1}}^{\infty} \Phi_3\big(s_n; x\big(g(s_n)\big)\big)\,\mathrm{d}s_n\cdots\mathrm{d}s_1 \\ := x(t_1) + \Psi_3\big(t; x\big(g(t)\big)\big) \quad \text{for} \quad t \ge t_1. \end{aligned}$$

Now it is easy to show the existence of a positive solution to the integral equation

$$\text{(6.9)}\qquad y_i(t) = c + \Psi_i(t, y_i[g(t)]) \quad \text{for} \quad t \ge t_1 \quad \text{and} \quad i = 1,2,3,$$

where $c = x(t_1)$.

We define $y_{i,n}(t)$, $i = 1, 2, 3$ and $n = 0, 1, \dots,$ as

$$y_{i,0}(t) = x(t)$$
$$y_{i,n+1}(t) = \begin{cases} c + \Psi_i\big(t, y_{i,n}\big(g(t)\big)\big) & \text{for } t \geq t_1 \text{ and } i = 1, 2, 3 \\ c \text{ for } t_0 \leq t \leq t_1. \end{cases}$$

Thus $y_{i,n}(t)$ is well–defined and for $t \geq t_1$, $i = 1, 2, 3$ and $n = 1, 2, \dots,$ we get

$$0 < y_{i,n}(t) \leq x(t), \quad c \leq y_{i,n+1}(t) \leq y_{i,n}(t).$$

By Lebesgue's Monotone Convergence Theorem there exists $y_i(t)$ such that $y_i(t) = \lim_{n\to\infty} y_{i,n}(t)$ for $t \geq t_1$ and

$$y_i(t) = c + \Psi_i(t, y_i[g(t)]) \text{ for } t \geq t_1.$$

It is easy to verify that $y_i(t)$ is a solution of equation (1.1) for $t \geq t_1$ and $i = 1, 2, 3$.

Next we employ Theorem 6.1 to extend the results obtained to the neutral differential equation

$$L_{2n}(x(t) + p(t)x\big(\sigma(t)\big) + q(t)f\big(x\big(g(t)\big)\big) = 0, \tag{6.10}$$

where the operator L_{2n} and the functions g, f and q are as in equation (1.1), and (v) $p(t)$ and $\sigma(t) \in C\big([t_0, \infty), \mathbb{R}\big)$, $\sigma'(t) > 0$ for $t \geq t_0$ and $\lim_{t\to\infty} \sigma(t) = \infty$.

In fact we prove the following comparison results.

Theorem 6.2. *Let conditions* (1.2) *and* (4.19) *hold*, $0 \leq p(t) \leq 1$, $p(t) \not\equiv 0$ *or* $p(t) \not\equiv 1$ *eventually, and* $\sigma(t) < t$ *for* $t \geq t_0$. *If the equation*

$$L_{2n}y(t) + q(t)f\big(1 - p\big(g(t)\big)\big)f\big(y\big(g(t)\big)\big) = 0 \tag{6.11}$$

is oscillatory, then equation (6.10) *is oscillatory.*

Theorem 6.3. *Let conditions* (1.2) *and* (4.19) *hold*, $p(t) \geq 1$, $p(t) \not\equiv 1$ *eventually and* $\sigma(t) > t$ *for* $t \geq t_0$. *If the equation*

$$L_{2n}z(t) + q(t)f\big(p^*\big(g(t)\big)\big)\, f\big(z\big(\sigma^{-1} \circ g(t)\big)\big) = 0, \tag{6.12}$$

where

$$p^*(t) = \frac{1}{p\big(\sigma^{-1}(t)\big)}\left(1 - \frac{1}{p\big(\sigma^{-1} \circ \sigma^{-1}(t)\big)}\right) \quad \textit{for } t \geq t_0$$

and σ^{-1} *is the inverse function of* σ, *is oscillatory, then equation* (6.10) *is oscillatory.*

Proofs of Theorems 6.2 and 6.3. Let $x(t)$ be a nonoscillatory solution of equation (6.10), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. Set $y(t) = x(t) + p(t)x\big(\sigma(t)\big)$. Then equation (6.10) becomes

$$L_{2n}y(t) + q(t)f\big(x\big(g(t)\big)\big) = 0 \text{ for } t \geq t_0. \tag{6.13}$$

It is easy to check that there exists a $t_1 \geq t_0$ such that

$$y(t) > 0 \quad \text{and} \quad y'(t) > 0 \quad \text{for} \quad t \geq t_1. \tag{6.14}$$

Next we assume that $0 \leq p(t) \leq 1$ and $\sigma(t) < t$ for $t \geq t_0$. Now

$$\begin{aligned} x(t) &= y(t) - p(t)x\big(\sigma(t)\big) \\ &= y(t) - p(t)\big(y\big(\sigma(t)\big) - p\big(\sigma(t)\big)x\big(\sigma \circ \sigma(t)\big)\big) \\ &\geq y(t) - p(t)y\big(\sigma(t)\big) \geq (1 - p(t))y(t) \quad \text{for} \quad t \geq t_1. \end{aligned} \tag{6.15}$$

Using (6.15) and (4.19) in equation (6.13) we have

$$L_{2n}y(t) + q(t)f\big(1 - p\big(g(t)\big)\big)\, f\big(y\big(g(t)\big)\big) \leq 0 \quad \text{for} \quad t \geq t_1. \tag{6.16}$$

Next we assume that $p(t) \geq 1$ and $\sigma(t) > t$ for $t \geq t_0$. Now

$$\begin{aligned} x(t) &= \frac{1}{p\big(\sigma^{-1}(t)\big)}\Big(y\big(\sigma^{-1}(t)\big) - x\big(\sigma^{-1}(t)\big)\Big) \\ &= \frac{y\big(\sigma^{-1}(t)\big)}{p\big(\sigma^{-1}(t)\big)} - \frac{1}{p\big(\sigma^{-1}(t)\big)}\left(\frac{y\big(\sigma^{-1} \circ \sigma^{-1}(t)\big)}{p\big(\sigma^{-1} \circ \sigma^{-1}(t)\big)} - \frac{x\big(\sigma^{-1} \circ \sigma^{-1}(t)\big)}{p\big(\sigma^{-1} \circ \sigma^{-1}(t)\big)}\right) \\ &\geq \frac{y\big(\sigma^{-1}(t)\big)}{p\big(\sigma^{-1}(t)\big)} - \frac{y\big(\sigma^{-1} \circ \sigma^{-1}(t)\big)}{p\big(\sigma^{-1}(t)\big)p\big(\sigma^{-1} \circ \sigma^{-1}(t)\big)} \\ &\geq \frac{1}{p\big(\sigma^{-1}(t)\big)}\left(1 - \frac{1}{p\big(\sigma^{-1} \circ \sigma^{-1}(t)\big)}\right))y\big(\sigma^{-1}(t)\big) \\ &:= p^*(t)y\big(\sigma^{-1}(t)\big) \quad \text{for} \quad t \geq t_1. \end{aligned} \tag{6.17}$$

Using (6.17) and (4.19) in equation (6.12) we obtain

$$L_{2n}y(t) + q(t)f\big(f^*\big(g(t)\big)\big)\, f\big(y\big(\sigma^{-1} \circ g(t)\big)\big) \leq 0 \quad \text{for} \quad t \geq t_1. \tag{6.18}$$

Inequalities (6.16) and (6.18) have eventually positive solutions and so by Theorem 6.1 equations (6.11) and (6.12) have also eventually positive solutions, which contradicts the hypotheses and completes the proof. □

Next we extend the results obtained to equation (1.1) when the function f need not be monotonic.

We need the following notations and a Lemma due to Mahfoud [**16**].

$$\mathbb{R}_{t_0} = \begin{cases} (-\infty, -t_0] \cup [t_0, \infty) & \text{if} \ \ t_0 > 0 \\ (-\infty, 0) \cup (0, \infty) & \text{if} \ \ t_0 = 0 \end{cases}$$

and

$$C_B(\mathbb{R}_{t_0}) = \{f \in C(\mathbb{R}) : f \text{ is of bounded variation on any interval } [a, b] \subset \mathbb{R}_{t_0}\}.$$

Lemma 6.1. *Suppose* $t_0 \geq 0$ *and* $f \in C(\mathbb{R})$. *Then* $f \in C_B(\mathbb{R}_{t_0})$ *if and only if* $f(x) = H(x)G(x)$ *for all* $x \in \mathbb{R}$, *where* $G : \mathbb{R}_{t_0} \to \mathbb{R}^+$ *is nondecreasing on* $(-\infty, -t_0)$ *and nonincreasing on* (t_0, ∞) *and* $H : \mathbb{R}_{t_0} \to \mathbb{R}$ *is nondecreasing on* $\mathbb{R}_{t_0}$.

Now we prove the following result.

Theorem 6.4. *Let condition* (1.2) *hold and assume that* $f \in C_B(\mathbb{R}_{t_0})$, $t_0 \geq 0$ *and let the functions* G *and* H *be a pair of continuous components of* f *with* H *being the nondecreasing one. If for all large* T *with* $g(t) > T$ *and all constant* $c > 0$, *the equation*

$$L_{2n}x(t) + q(t)G\big(c\phi\big(g(t), T; a\big)\big)H\big(x\big(g(t)\big)\big) = 0 \tag{6.19}$$

is oscillatory, where the function ϕ *is as in* (3.13), *then equation* (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.1), say, $x(t) > 0$ for $t \geq t_0 \geq 0$. As in the proof of Theorem 3.3 we obtain (3.15) for $t \geq T_1$. There exists a $T_2 \geq T \geq T_1$ such that $g(t) > T$ and

$$x\big(g(t)\big) \geq b\phi\big(g(t), T; a\big) \quad \text{for} \quad t \geq T. \tag{6.20}$$

Using (6.20) in equation (1.1) we have

$$\begin{aligned} -L_{2n}x(t) &= q(t)f\big(x\big(g(t)\big)\big) = q(t)G\big(x\big(g(t)\big)\big)H\big(x\big(g(t)\big)\big) \\ &\geq q(t)G\big(b\phi\big(g(t), T; a\big)\big)H\big(x\big(g(t)\big)\big) \quad \text{for} \quad t \geq T_2. \end{aligned} \tag{6.21}$$

The inequality (6.21) has an eventually positive solution and so by Theorem 6.1 equation (6.19) has also an eventually positive solution, which contradicts the hypotheses and completes the proof. ☐

As examples of functions $f(x)$ which are not monotonic we give the following:

(i) $f(x) = \dfrac{|x|^{\beta-1}x}{1 + |x|^{\gamma}}$, where β and γ are positive constants,

(ii) $f(x) = |x|^{\beta-1}x \exp(-|x|^{\gamma})$, where β and γ are positive constants,

(iii) $f(x) = |x|^{\beta-1}x \operatorname{sech} x$, where β is a positive constant.

We may note that the above results are not applicable to equation (1.1) with any one of the above choices of f.

REMARKS.

1. The results of this paper are presented in a form which is essentially new and of a higher degree of generality. In fact one can easily extract more criteria than those presented for the oscillation of equation (1.1) and/or related equations. The formulation of such criteria is left to the reader.
2. The results of this paper may be extended to forced equations of the form

$$L_{2n}x(t) + q(t)f\big(x\big(g(t)\big)\big) = e(t),$$

where $e \in C\big([t_0, \infty), \mathbb{R}\big)$.

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(Received May 24, 2007)
(Revised November 7, 2007)

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