Applicable Analysis and Discrete Mathematics, **1** (2007), 335–339.
Available electronically at **http://pefmath.etf.bg.ac.yu**

SOME EASY TO REMEMBER ABSTRACT FORMS OF EKELAND'S VARIATIONAL PRINCIPLE AND CARISTI'S FIXED POINT THEOREM

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If X is a set, then an extended real-valued function Φ of X^2 is called an écart on X. Moreover, if d and Φ are écarts on X, then for any $x, y \in X$ we write $x \leq y$ if and only if $d(x, y) \leq \Phi(x, y)$. Thus, we call the pair (d, Φ) admissible at a point $a \in X$ if X has a maximal element b with $a \leq b$. Here, an important particular case is when d is a certain metric on X and Φ is of the form $\Phi(x, y) = \varphi(x) - \psi(y)$.
These definitions allow us to easily state and prove some easily remembered abstract forms of Ekeland's variational principle and Caristi's fixed point theorem. For instance, we show that if F is a relation on X and $a \in X$ such that there exists a pair (d, Φ) of écarts on X which is admissible at a and satisfies $d(x, y) \leq \Phi(x, y)$ for all $x \in X$ and $y \in F(x)$, then there exists $b \in X$, with $d(a, b) \leq \Phi(a, b)$, such that $F(b) \subset \{b\}$.

1. SOME GENERAL DEFINITIONS

Definition 1.1. *If X is a set, then an extended real-valued function Φ of X^2 is called an écart on X.*

Example 1.2. If φ and ψ are functions of X into $\mathbb{R}$, then the function Φ, defined by $\Phi(x, y) = \varphi(x) - \psi(y)$ for all $x, y \in X$, is a natural écart on X.

Definition 1.3. *A set X equipped with a relation $\leq$ is called a goset (generalized ordered set).*

Remark 1.4. An element x of the goset X is called maximal if $x \leq y$ implies $x = y$ for all $y \in X$

2000 Mathematics Subject Classification. 06A06, 54H25.
Keywords and Phrases: Ordered sets, maximal elements, variational principles, fixed point theorems.

Definition 1.5. *If d and Φ are écarts on a set X, then the relation $\leq$, defined such that for any $x, y \in X$ we have*

$$x \leq y \Leftrightarrow d(x, y) \leq \Phi(x, y),$$

will be called the inequality relation induced by d and Φ.

REMARK 1.6. Thus, the induced inequality relation $\leq$ is always total on X in the sense that for any $x, y \in X$ we have either $x \leq y$ or $y \leq x$.

Moreover, it can be easily seen that if d satisfies the triangle inequality and Φ satisfies the converse of that inequality, then the induced inequality relation is transitive.

Note that if in particular Φ is as in Example 1.2 and $\varphi \leq \psi$, then Φ already satisfies the converse of the triangle inequality.

Definition 1.7. *A pair (d, Φ) of écarts on a set X will be called admissible at a point $a \in X$ if X, equipped with the inequality relation $\leq$ induced by d and Φ, has a maximal element b with $a \leq b$.*

REMARK 1.8. Some sufficient conditions on $\leq$ and Φ, in order that the pair (d, Φ) be admissible at a, have been given in our former paper [**3**].

Now, as an immediate consequence of the above definitions, we can state

Theorem 1.9. *If X is a goset and $a \in X$ such that there exists a pair (d, Φ) of écarts on X which is admissible at a and satisfies*

$$d(x, y) \leq \Phi(x, y)$$

for all $x, y \in X$, with $x \leq y$, then there exists a maximal element b of X with $d(a, b) \leq \Phi(a, b)$.

Proof. Denote now by $\preccurlyeq$ the inequality relation induced by d and Φ. Then, by Definition 1.7, there exists $b \in X$, with $a \preccurlyeq b$, such that b is a maximal element of X with respect to $\preccurlyeq$.

By Definition 1.5, $a \preccurlyeq b$ implies $d(a, b) \leq \Phi(a, b)$. Moreover, from the second part of the hypothesis of the theorem, by Definition 1.5, we can see that $x \leq y$ implies $x \preccurlyeq y$ for all $x, y \in X$. Hence, it is clear that b is a maximal element of X with respect to $\leq$ too. Namely, if $y \in X$ such that $b \leq y$, then $b \preccurlyeq y$, and thus $b = y$ also holds. □

2. ABSTRACT FORMS OF EKELAND'S AND CARISTI'S THEOREMS

By using the above definitions, we can also easily state and prove an abstract form of a simplified version of EKELAND's variational principle [**2**].

Theorem 2.1. *If d and Φ are écarts on a set X and $a \in X$ such that the pair (d, Φ) is admissible at a, then there exists $b \in X$ such that*

$$d(a, b) \leq \Phi(a, b) \quad \textit{and} \quad \Phi(b, y) < d(b, y)$$

for all $y \in X$ *with* $y \neq b$.

Proof. Consider X to be equipped with the inequality relation $\leq$ induced by d and Φ. Then, by Definition 1.7, there exists a maximal element b of X with $a \leq b$. This implies that $b \not\leq y$ for all $y \in X$ with $y \neq b$. Hence, by Definition 1.5, it is clear that $d(a,\, b) \leq \Phi(a,\, b)$ and $d(b,\, y) \not\leq \Phi(b,\, y)$, i.e., $\Phi(b,\, y) < d(b,\, y)$ for all $y \in X$ with $y \neq b$. □

Remark 2.2. If in particular Φ is as in Example 1.2, then the conclusion of the above theorem gives only that

$$\psi(b) + d(a,\, b) \leq \varphi(a) \quad \text{and} \quad \varphi(b) < \psi(y) + d(b,\, y)$$

for all $y \in X$ with $y \neq b$.

Now, as an abstract form of a localized version of Caristi's fixed point theorem [**1**],we can also easily state and prove the following

Theorem 2.4. *If* f *is a function of a set* X *into itself and* $a \in X$ *such that there exists a pair* $(d,\, \Phi)$ *of écarts on* X *which is admissible at* a *and satisfies*

$$d\big(x,\, f(x)\big) \leq \Phi\big(x,\, f(x)\big)$$

for all $x \in X$, *then there exists* $b \in X$ *such that*

$$d(a,\, b) \leq \Phi(a,\, b) \quad \textit{and} \quad f(b) = b.$$

Proof. By Theorem 2.1, there exists $b \in X$ such that $d(a,\, b) \leq \Phi(a,\, b)$ and $\Phi(b,\, y) < d(b,\, y)$ for all $y \in X$ with $y \neq b$.

Hence, it is clear $f(b) = b$. Namely, otherwise we would have $\Phi\big(b,\, f(b)\big) < d\big(b,\, f(b)\big)$ which would contradict the assumption of the theorem. □

Remark 2.5. If in particular Φ is as in Example 1.2, then the assumption of the above theorem means only that

$$d\big(x,\, f(x)\big) \leq \varphi(x) - \psi\big(f(x)\big)$$

for all $x \in X$.

3. SET-VALUED EXTENSIONS OF THE ABSTRACT CARISTI THEOREM

As an immediate consequence of Theorem 2.4, we can state the following more general

Theorem 3.1. *If* F *is a relation on a set* X *and* $a \in X$ *such that there exists a pair* $(d,\, \Phi)$ *of écarts on* X *which is admissible at* a *and for each* $x \in X$ *there exists* $y \in F(x)$ *such that*

$$d(x,\, y) \leq \Phi(x,\, y),$$

then there exists $b \in X$ *such that*

$$d(a,\, b) \leq \Phi(a,\, b) \quad \textit{and} \quad b \in F(b).$$

Proof. By the Axiom of Choice, there exists a function f of X into itself such that

$$f(x) \in F(x) \quad \text{and} \quad d\big(x,\, f(x)\big) \leq \Phi\big(x,\, f(x)\big)$$

for all $x \in X$. Therefore, by Theorem 2.4, there exists $b \in X$ such that $d(a,\, b) \leq \Phi(a,\, b)$ and $f(b) = b$. Thus, in particular $b = f(b) \in F(b)$ also holds. □

By using Theorem 2.1, we can also easily prove the following extension of Theorem 2.4.

Theorem 3.2. *If F is a relation on a set X and $a \in X$ such that there exists a pair $(d,\, \Phi)$ of écarts on X which is admissible at a and satisfies*

$$d(x,\, y) \leq \Phi(x,\, y)$$

for all $x \in X$ and $y \in F(x)$, then there exists $b \in X$ such that

$$d(a,\, b) \leq \Phi(a,\, b) \quad \textit{and} \quad F(b) \subset \{b\}.$$

Proof. By Theorem 2.1, there exists $b \in X$ such that $d(a,\, b) \leq \Phi(a,\, b)$ and $\Phi(b,\, y) < d(b,\, y)$ for all $y \in X$ with $y \neq b$.

Hence, it is clear that if $y \in F(b)$, then $y = b$. Namely, otherwise we would have $\Phi(b,\, y) < d(b,\, y)$, which would contradict the assumption of the theorem. □

Remark 3.3. It is noteworthy that Theorem 2.1 can already be derived from the functional particular case of Theorem 3.2.

Namely, if the conclusion of Theorem 2.1 does not hold, then by taking

$$D = \big\{ x \in X : d(a,\, x) \leq \Phi(a,\, x) \big\}$$

and using the Axiom of Choice we can get a function f of D into X such that

$$f(x) \neq x \quad \text{and} \quad d\big(x,\, f(x)\big) \leq \Phi\big(x,\, f(x)\big)$$

for all $x \in D$. Therefore, by Theorem 3.2, there exists $b \in D$ such that $f(b) \subset \{b\}$, and thus $f(b) = b$, which is now a contradiction. □

Moreover, it is also noteworthy that the following extension of Theorem 2.4 can also be immediately derived from Theorem 3.2.

Theorem 3.4. *If $\mathcal{F}$ is a family of functions of a set X into itself and $a \in X$ such that there exists a pair $(d,\, \Phi)$ of écarts on X which is admissible at a and satisfies*

$$d\big(x,\, f(x)\big) \leq \Phi\big(x,\, f(x)\big)$$

for all $x \in X$ and $f \in \mathcal{F}$, then there exists $b \in X$ such that

$$d(a,\, b) \leq \Phi(a,\, b) \quad \textit{and} \quad f(b) = b$$

for all $f \in \mathcal{F}$.

Proof. Define $F = \bigcup \mathcal{F}$. Then, F is a relation on X such that the conditions of Theorem 3.2 hold. Therefore, there exists $b \in X$ such that $d(a, b) \leq \Phi(a, b)$ and $F(b) \subset \{b\}$. Hence, since $F(b) = \big\{ f(b) \big\}_{f \in \mathcal{F}}$, it is clear that $f(b) = b$ for all $f \in \mathcal{F}$. $\square$

Remark 3.5. In this respect, it is also noteworthy that Theorem 1.9 can also be immediately derived from Theorem 3.2.

Namely, if the conditions of Theorem 1.9 hold, then by taking $F = \leq$ we can at once see that the conditions of Theorem 3.2 hold. Therefore, there exists $b \in X$, with $d(a, b) \leq \Phi(a, b)$, such that $F(b) \subset \{b\}$. Hence, it is clear that if $y \in X$ such that $b \leq y$, i.e., $y \in F(b)$, then we necessarily have $b = y$. Therefore, b is maximal.

The extensive References of the original manuscript have been dramatically reduced according to the requirements of the editors and referees. The original References can be obtained from the author by E–mail.

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(Received January 28, 2007)