Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 18 (2006), 59–63.
Available electronically at **http: //pefmath.etf.bg.ac.yu**

THERE ARE INTEGRAL TREES OF DIAMETER 7

Pavel Híc, Milan Pokorný

A graph G is called integral if all the roots of the characteristic polynomial $P(G;x)$ are integers. In this paper, four integral trees of diameter 7 are given. These integral trees are the first integral trees of diameter 7 which have been found so far.

1. INTRODUCTION

The notion of integral graphs was introduced by F. Harary and A. J. Schwenk in 1974 (see [**4**]). A graph G is called integral if all the zeros of the characteristic polynomial $P(G;x)$ are integers. In general, the problem of characterizing integral graphs seems to be very difficult. Thus, it makes sense to restrict our investigations to some families of graphs. Trees represent one important family of graphs, for which the problem has been considered. Results on integral trees with diameter k, where $2 \leq k \leq 10$, can be found in [**1, 5–15**]. From these papers follows that integral trees of diameters 1, 2, 3, 4, 5, 6, 8, and 10 can be constructed. But no integral tree of diameter 7 or 9 was found in these papers. In [**5**] it is proved that there is no balanced integral tree of diameter 7 and diameter $4k+1$ for $k \geq 1$. In [**15**; Question 7], the following question was given: "Are there any integral trees of diameter 7?" In this paper four integral trees of diameter 7 are given.

From [**5**; Theorem 4.6] follows that an integral tree of diameter 7 cannot be balanced. A tree T is called balanced if the vertices at the same distance from the center of T have the same degree. We shall code a balanced tree of diameter $2k$ by the sequence $(n_k, n_{k-1}, \dots, n_1)$ or the tree $T(n_k, n_{k-1}, \dots, n_1)$, where n_j $(j = 1, 2, \dots, k)$ denotes the number of successors of a vertex at the distance $k-j$ from the center. Let the tree $T(n_k, n_{k-1}, \dots, n_1)\Theta T(m_j, m_{j-1}, \dots, m_1)$ be obtained by joining the center w of $T(n_k, n_{k-1}, \dots, n_1)$ and the center v of $T(m_j, m_{j-1}, \dots, m_1)$

2000 Mathematics Subject Classification: 05C50,05C05
Keywords and Phrases: Integral tree, diameter, divisor, characteristic polynomial.

with a new edge. This tree is denoted by $T(n_1, n_2, \ldots, n_{k-1}, n_k; 1; m_j, m_{j-1}, \ldots, m_1)$. Clearly, if $n_i \neq m_i$ for any i, the tree is not balanced. In this paper we investigate trees of diameter 7 which can be obtained from two balanced trees of diameter 6 by operation Θ.

In [**5**] the following Theorem is proved:

Theorem A (see [**5**; Theorem 2.1.]). *Let D be a front-divisor of a tree T and C be the corresponding codivisor. Then $P(C;x) = \prod_{T_i \subseteq T - V(D)} P(T_i;x)$, where T_i are connected components of $T - V(D)$ and in the case that D has the v_1-based loop, T_1 is the connected component of $T - V(D)$ with the u_1-based loop valued by -1. (u_1 is from the center of T).*

Note. It follows from this theorem that the characteristic polynomial $P(C;x)$ of a codivisor of a tree T can be expressed in terms of proper subtrees of T. The properties of front-divisors and corresponding codivisors of trees and graphs are discussed in [**2, 5, 8**].

2. INTEGRAL TREES OF DIAMETER 7

Theorem 1. *Let $T(n_3, n_2, n_1)\Theta T(k_3, k_2, k_1) = T(n_1, n_2, n_3; 1; k_3, k_2, k_1)$ be a tree of diameter 7. Then $T(n_1, n_2, n_3; 1; k_3, k_2, k_1)$ is integral if and only if the following statements hold*:

1. *The corresponding divisor $D(n_1, n_2, n_3; 1; k_3, k_2, k_1)$ is integral.*
2. *$T(n_2, n_1), T(k_2, k_1), T(n_1)$ and $T(k_1)$ are integral.*

Proof. It is easy to verify (see [**2**, **5**, or **8**]) that the corresponding front-divisor $D(n_1, n_2, n_3; 1; k_3, k_2, k_1)$ of the tree $T = T(n_1, n_2, n_3; 1; k_3, k_2, k_1)$ has characteristic polynomial

$$\begin{aligned} P(D;x) \ &= \ x^8 - (n_1 + n_2 + n_3 + k_1 + k_2 + k_3 + 1)x^6 \\ &+ ((n_1 + n_2 + n_3)(k_1 + k_2 + k_3) + n_1 n_3 + n_1 + n_2 + k_1 k_3 + k_1 + k_2)x^4 \\ &+ (n_1 n_3 (k_3 + k_2 + k_1) + k_1 k_3 (n_3 + n_2 + n_1) + (n_1 + n_2)(k_1 + k_2))x^2 \\ &+ n_1 n_3 k_1 k_3. \end{aligned}$$

Clearly, the connected components of $T - V(D)$ have the form $T(n_2, n_1)$, $T(k_2, k_1)$, $T(n_1)$, $T(k_1)$, and isolated vertices. The corresponding codivisor

$$C(n_1, n_2, n_3; 1; k_3, k_2, k_1)$$

of the tree $T = T(n_1, n_2, n_3; 1; k_3, k_2, k_1)$ has characteristic polynomial

$$\begin{aligned} P(C;x) = x^{(k_1-1)(k_2-1)k_3 + (n_1-1)(n_2-1)n_3 + (k_1-1) + (n_1-1) + (k_3-1) + (n_3-1)} \\ \cdot \left(x^2 - n_1\right)^{(n_2-1)n_3} \cdot \left(x^2 - k_1\right)^{(k_2-1)k_3} \\ \cdot \left(x^2 - (n_1 + n_2)\right)^{n_3-1} \cdot \left(x^2 - (k_1 + k_2)\right)^{k_3-1}. \end{aligned}$$

Because of $P(T;x) = P(D;x).P(C;x)$, the tree T is integral if and only if the polynomials $P(D;x)$ and $P(C;x)$ have only integer zeros. The proof is finished.

It is known that $T(n_2,n_1), T(k_2,k_1), T(n_1), T(k_1)$ are integral if and only if $n_1, k_1, n_2+n_1, k_2+k_1$ are squares. Using computers we investigated all the trees $T(n_1,n_2,n_3;1;k_3,k_2,k_1)$, where n_1 and k_1 are squares, n_1+n_2 and k_1+k_2 are squares, $n_1+n_2 \leq 10000$, $k_1+k_2 \leq 10000$, $n_3 < 1200$, $k_3 < 1200$. From all these trees only four are integral. Their list can be found in example 1–4. In these examples $n_3 = k_3$. That is why we investigated all the trees $T(n_1,n_2,n_3;1;k_3,k_2,k_1)$, where n_1 and k_1 are squares, n_1+n_2 and k_1+k_2 are squares, $n_1+n_2 \leq 10000$, $k_1+k_2 \leq 10000$, $1200 \leq n_3 = k_3 < 3000$, but we have found no integral tree among them.

Example 1. The tree

$$T(49,480,270;1;270,420,64) = T(270,480,49)\Theta T(270,420,64)$$

is integral.

The characteristic polynomial of its divisor is

$$P(D(49,480,270;1;270,420,64);x) = x^8 - 1554x^6 + 633969x^4 - 24038176x^2 + 228614400.$$

The spectrum of the divisor is $Spec(D) = \{\pm 4, \pm 5, \pm 27, \pm 28\}$.

The characteristic polynomial of the corresponding codivisor is

$$P(C;x) = x^{13335679}\left\{\left(x^2-484\right)\left(x^2-529\right)\right\}^{269}\left(x^2-64\right)^{113130}\left(x^2-49\right)^{129330}.$$

The spectrum of the tree is

$$Spec(T) = \begin{pmatrix} \pm 28 & \pm 27 & \pm 23 & \pm 22 & \pm 8 & \pm 7 & \pm 5 & \pm 4 & 0 \\ 1 & 1 & 269 & 269 & 113130 & 129330 & 1 & 1 & 13335679 \end{pmatrix}.$$

Example 2. The tree

$$T(25,264,504;1;504,220,36) = T(504,264,25)\Theta T(504,220,36)$$

is integral.

The characteristic polynomial of its divisor is

$$P(D(25,264,504;1;504,220,36);x) = x^8 - 1554x^6 + 633969x^4 - 24038176x^2 + 228614400.$$

The spectrum of the divisor is $Spec(D) = \{\pm 4, \pm 5, \pm 27, \pm 28\}$.

The characteristic polynomial of the corresponding codivisor is

$$P(C;x) = x^{7045473}\left(x^2-36\right)^{110376}\left(x^2-25\right)^{132552}\left\{\left(x^2-256\right)\left(x^2-289\right)\right\}^{503}$$

.

The spectrum of the tree is

$$Spec(T) = \begin{pmatrix} \pm 28 & \pm 27 & \pm 17 & \pm 16 & \pm 6 & \pm 5 & \pm 4 & 0 \\ 1 & 1 & 503 & 503 & 110376 & 132553 & 1 & 7045473 \end{pmatrix}.$$

Example 3. The tree

$$T(3136, 5328, 1140; 1; 1140, 5700, 3136) = T(1140, 5328, 3136)\Theta T(1140, 5700, 3136)$$

is integral.

The characteristic polynomial of its divisor is

$$P(D(3136, 5328, 1140; 1; 1140, 5700, 3136); x) = x^8 - 19581x^6 + 102976884x^4 - 70074071104x^2 + 12780911001600.$$

The spectrum of the divisor is $Spec(D) = \{\pm 19, \pm 20, \pm 96, \pm 98\}$.

The characteristic polynomial of the corresponding codivisor is

$$P(C; x) = x^{39405829948} \left(x^2 - 3136\right)^{12569640} \left\{\left(x^2 - 8836\right)\left(x^2 - 8464\right)\right\}^{1139}$$

.

The spectrum of the tree is

$$Spec(T) = \begin{pmatrix} \pm 98 & \pm 96 & \pm 94 & \pm 92 & \pm 56 & \pm 20 & \pm 19 & 0 \\ 1 & 1 & 1139 & 1139 & 12569640 & 1 & 1 & 39405829948 \end{pmatrix}.$$

Example 4. The tree

$$T(625, 4704, 1188; 1; 1188, 4900, 576) = T(1188, 4704, 625)\Theta T(1188, 4900, 576)$$

is integral.

The characteristic polynomial of its divisor is

$$P(D(625, 4704, 1188; 1; 1188, 4900, 576); x) = x^8 - 13182x^6 + 44866881x^4 - 9436706500x^2 + 508083840000.$$

The spectrum of the divisor is $Spec(D) = \{\pm 10, \pm 11, \pm 80, \pm 81\}$.

The characteristic polynomial of the corresponding codivisor is

$$P(C; x) = x^{6832900809} \left(x^2 - 625\right)^{5587164} \left(x^2 - 576\right)^{5820012} \cdot \left(\left(x^2 - 5476\right)\left(x^2 - 5329\right)\right)^{1187}.$$

The spectrum of the tree is

$$Spec(T) = \begin{pmatrix} \pm 81 & \pm 80 & \pm 74 & \pm 73 & \pm 25 & \pm 24 & \pm 11 & \pm 10 & 0 \\ 1 & 1 & 1187 & 1187 & 5587164 & 5820012 & 1 & 1 & 6832900809 \end{pmatrix}.$$

Acknowledgements. The authors wish to express their thanks to the referees for their help in improving the quality of this paper.

Research partially supported by VEGA grant 1/1001/04.

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Department of Mathematics and Computer Science,
Faculty of Education; Trnava University,
Priemyselná 4; 918 43 Trnava; Slovakia
E–mail: phic@truni.sk; mpokorny@truni.sk

(Received September 22, 2006)