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SOLUTION OF AN INTEGRAL IMPORTANT IN THE ANALYSIS OF WIRE ANTENNAS ABOVE GROUND

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Analytical solution of an integral important in the analysis of wire antennas above ground is presented in detail. The integral plays an important role in numerical analysis of wire antennas, because for small source-to-field distances it is quasi-singular and thus very demanding for numerical integration. Therefore, its exact analytical solution is preferred.

1. INTRODUCTION

1.1. Source of the integral

In numerical analysis of wire antennas above real ground [**1**], [**2**] by the exact approach (i.e., by the use of Sommerfeld integrals [**3**]), the term

$$g = \frac{X}{R(R+Z)} \tag{1}$$

appears in one of components of electromagnetic field potentials [**4**]–[**6**]. Here (referring to Figure 1), R is the distance of the field point (F) from the source point (S), Z is projection of the vector $\overrightarrow{\mathrm{SF}}$ on vertical axis Z_s and X is its projection on horizontal axis X_s. The horizontal interface plane divides the space into two half-spaces, source point being always in the lower ($Z_s \leq 0$) and field point being always in the upper ($Z_s \geq 0$) half-space. (Source point (S) is the 'image' of the original source point with respect to the interface plane as the plane of symmetry.)

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This component of potential due to current $I(x)$ along the straight antenna segment (Figure 1) is given by

$$\int_{x_{01}}^{x_{02}} I(x)g(x)\,\mathrm{d}x, \tag{2}$$

where now $Z = ax + b$, $X = px + q$, $R = \sqrt{x^2 + d^2}$ and d is the normal distance of point F from the antenna segment axis (x-axis). Very often antenna current is approximated by polynomial with unknown coefficients to be determined during the antenna numerical solution. Thus, integrals of the type

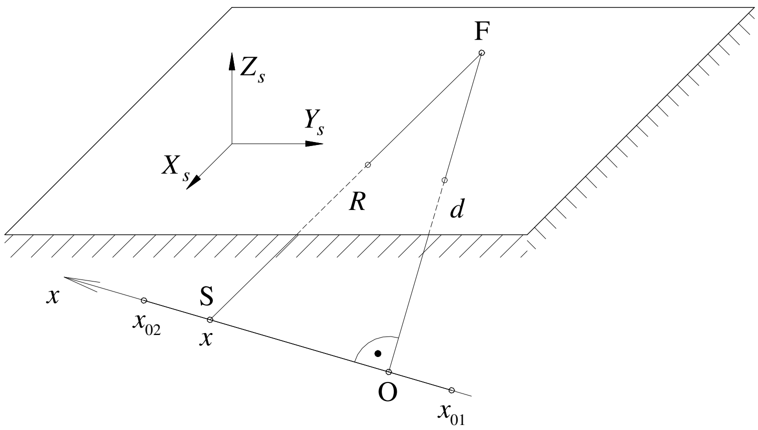

Figure 1. Geometry of the problem.

$$\int_{x_{01}}^{x_{02}} x^i \frac{1}{\sqrt{x^2 + d^2}\left(\sqrt{x^2 + d^2} + ax + b\right)}\,\mathrm{d}x, \qquad i = 0, 1, 2, \ldots \tag{3}$$

need to be evaluated. For field point F close to the antenna segment, those integrals are "quasi-singular" in the sense that values of the integrand or its derivatives become very large around $x = 0$. (The integration path in the complex x-plane passes close to singular points, $x_{\mathrm{s}} = \pm\mathrm{i}d$, of the function g.) The quasi-singular behavior is pronounced the most for small values of i ($i = 0, 1, 2$). As i gets larger the quasi-singularity is less pronounced, being moved to derivatives of the integrand. Therefore, an analytical solution of the integral (3) for smaller values of i could be most beneficial for efficient and accurate overall antenna solution. In this paper, solution of (3) for $i = 0, 1, 2$ will be derived.

1.2. Restrictions of the integral parameters

Due to geometry presented in Figure 1, the following restrictions on the integral parameters d, a and b apply:

1. $d > 0$ or ($d = 0$ and $0 \notin [x_{01}, x_{02}]$) (The distance can not be negative, and the field point is not on the antenna segment.)
2. $|b| \leq d$ (The vertical distance between the two points can not be greater than their distance.)
3. $|a| \leq 1$ (The absolute value of the slope of antenna segment can not be greater than one.)
4. $a^2d^2 - d^2 + b^2 \leq 0$ (For the given slope of the line d, the slope of the antenna segment is limited.)
5. $ax_{01}+b \geq 0$ and $ax_{02}+b \geq 0$ (The vertical distance between the field point and the source point can not be negative, as they are in separate half-spaces.)

1.3. Initial comments

To solve the integral (3) analytically, we should find the indefinite integral,

$$F_i(x) = \int x^i \frac{1}{\sqrt{x^2+d^2}\left(\sqrt{x^2+d^2}+ax+b\right)}\,\mathrm{d}x, \qquad i = 0, 1, 2. \tag{4}$$

Restriction 4 enables introduction of a new parameter

$$L = \sqrt{-a^2d^2 + d^2 - b^2}, \quad L \geq 0. \tag{5}$$

This parameter will play an important role in solving the integral. Also, notation $F_i(x) = \left\{ \begin{array}{c} F_0(x) \\ F_1(x) \\ F_2(x) \end{array} \right\}$ will be used for brevity.

Depending on the parameter values, the integral (4) has some special cases that must be excluded from the general solution. Also, there are some special cases that, although being regularly included in the general solution, are much simpler. We will analyze the general solution and all these special cases.

2. THE 'TRIVIAL' SPECIAL CASE $d = 0$

Case $d = 0$, $0 \notin [x_{01}, x_{02}]$ is a case when the field point belongs to the segment line, and it can be considered the most simple one. From restriction 2 it follows that $b = 0$. Also $R = |x|$, $Z = ax$, and x does not change the sign on the integration path, so the integral reduces to a simple form,

$$F_i(x) = \int x^i \frac{1}{|x|(|x|+ax)}\,\mathrm{d}x = \frac{1}{1+a\,\mathrm{sgn}\,x} \left\{ \begin{array}{c} -1/x \\ \ln|x| \\ x \end{array} \right\} = \frac{1}{1+|a|} \left\{ \begin{array}{c} -1/x \\ \ln|x| \\ x \end{array} \right\}. \tag{6}$$

3. TRANSFORMATION OF THE INTEGRAND

For non-trivial cases ($d \neq 0$) a useful transformation of the integrand is multiplication of nominator and denominator by $(R - Z)$. Original integral, $F_i(x)$, breaks into two simpler integrals, $F_i^{(1)}(x)$ and $F_i^{(2)}(x)$,

$$(7) \quad F_i(x) = \int x^i \frac{R - Z}{RP^2}\,\mathrm{d}x = \int x^i \frac{1}{P^2}\,\mathrm{d}x - \int x^i \frac{Z}{RP^2}\,\mathrm{d}x = F_i^{(1)}(x) - F_i^{(2)}(x),$$

where

$$(8) \; P^2 = R^2 - Z^2 = (1 - a^2)x^2 - 2abx + (d^2 - b^2) = Ax^2 + Bx + C, \quad P \geq 0,$$

$$(9) \qquad A = 1 - a^2, \quad B = -2ab, \quad C = d^2 - b^2,$$

$$(10) \qquad F_i^{(1)}(x) = \int x^i \frac{1}{P^2}\,\mathrm{d}x = \int x^i \frac{1}{Ax^2 + Bx + C}\,\mathrm{d}x,$$

$$(11) \qquad F_i^{(2)} = \int x^i \frac{Z}{RP^2}\,\mathrm{d}x = \int x^i \frac{ax + b}{\sqrt{x^2 + d^2}(Ax^2 + Bx + C)}\,\mathrm{d}x.$$

Function $P^2(x)$ is important for solving non-trivial cases of the integral and it should be examined in more detail. First of all, from geometry it can be seen that P^2 is a square of a horizontal distance between the source and the field point, so it can not be negative.

Next, for some parameters' values it has special properties, leading to special cases of the integral.

First such case is $a = \pm 1$. From restriction 4 it follows that $b = 0$, so $P^2 = d^2$ ($P = d$), and the integral is greatly simplified.

For $a \neq \pm 1$, P^2 function is a quadratic function of x whose discriminant is

$$(12) \qquad D = B^2 - 4AC = 4(a^2d^2 - d^2 + b^2) = -4L^2.$$

For $L \neq 0$ it has a pair of complex roots,

$$(13) \qquad x_{1,2} = \frac{-B \pm \sqrt{D}}{2A} = \frac{ab \pm jL}{1 - a^2}.$$

For $L = 0$ it has a double real root,

$$(14) \qquad x_0 = \frac{ab}{1 - a^2},$$

and this is a second special case. Note that in all the three special cases mentioned: 1. $d = 0$, 2. $a = \pm 1$ and 3. $L = 0$, parameter L equals zero. It will be shown that these special cases are not a part of a general solution. Thus, they must be solved separately. The remaining two solutions are given in the next two sections. Case $L \neq 0$ is a general case of the integral.

Factorization of the function P^2 (for $d \neq 0$ and $a \neq \pm 1$) is the following:

$$\text{(15)}\quad P^2 = \begin{cases} A(x-x_1)(x-x_2) = A\Big(x - x_0 - j\,\dfrac{L}{A}\Big)\Big(x - x_0 + j\dfrac{L}{A}\Big), & (L \neq 0), \\ A(x-x_0)^2 = \dfrac{b^2}{d^2}\,(x-x_0)^2, & (L = 0) \end{cases}.$$

There are some cases that are common in practice and greatly simplifies the solution of the integral, although they are part of the general solution. One of these cases is $|b| = d \neq 0$ (only $b = +d$ is possible, due to restriction 5). From restriction 4 it follows $a = 0$. Another such case is $b = 0$, $a \neq \pm 1$. This reduces P^2 to $(1-a^2)x^2 + d^2$.

4. SPECIAL CASE $a = \pm 1$

For $a = \pm 1$ and $d \neq 0$, b is zero, $P^2 = d^2$ and the integral has the form

$$\text{(16)}\qquad F_i(x) = F_i^{(1)}(x) - F_i^{(2)} = \int \frac{x^i}{d^2}\,\mathrm{d}x - \int x^i \frac{ax}{d^2\sqrt{x^2+d^2}}\,\mathrm{d}x.$$

The first integral is a table one. The second integral can be reduced to table integrals by the change of variables, $\sqrt{x^2+d^2} = R$. Final result is

$$\text{(17)}\qquad F_i(x) = \frac{x^{i+1}}{d^2(i+1)} - \frac{a}{d^2}\begin{Bmatrix} R \\ \dfrac{xR - d^2\ln(x+R)}{2} \\ \dfrac{1}{3}R^3 - Rd^2 \end{Bmatrix}.$$

5. SPECIAL CASE $L = 0$

For $L = 0$ ($a \neq \pm 1$, $d \neq 0$) integrals $F_i^{(1)}(x)$ and $F_i^{(2)}(x)$ are

$$\text{(18)}\qquad F_i^{(1)}(x) = \int \frac{x^i}{A(x-x_0)^2}\,\mathrm{d}x,$$

$$\text{(19)}\qquad F_i^{(2)}(x) = \int x^i \frac{ax+b}{A\sqrt{x^2+d^2}\,(x-x_0)^2}\,\mathrm{d}x, \qquad x_0 = \frac{ad^2}{b}, \quad A = \frac{b^2}{d^2}.$$

Note that both integrals have singularity at $x = x_0$. However, in final solution singular terms should mutually cancel. Also, in this case $b \neq 0$, because from $b = 0$, $L = 0$ and $d \neq 0$ follows $a = \pm 1$, and that case was already analyzed.

In the first integral, change of variables $x - x_0 = t$ reduces it to table integrals. The result is

$$\text{(20)}\qquad F_i^{(1)}(x) = \frac{d^2}{b^2}\begin{Bmatrix} -\dfrac{1}{x-x_0} \\ \ln|x-x_0| - \dfrac{x_0}{x-x_0} \\ 2x_0\ln|x-x_0| + x - \dfrac{x_0^2}{x-x_0} \end{Bmatrix}.$$

For the second integral different techniques can be used for different values of i. For $i=0$, change of variables $\dfrac{\sqrt{x^2+d^2}}{x-x_0}=t$ directly solves the integral,

$$F_0^{(2)}(x) = -\frac{b}{Ad^2}\int \mathrm{d}t = -\frac{\sqrt{x^2+d^2}}{b(x-x_0)}. \tag{21}$$

For $i=1$ and $i=2$ Derive gives

$$F_1^{(2)}(x) = \frac{d^2}{b^2}\left(\ln\frac{|x-x_0|}{R+Z} + a\ln(x+R) - \frac{aR}{x-x_0}\right), \tag{22}$$

$$F_2^{(2)}(x) = \frac{d^2}{b^2}\left(2x_0\ln\frac{|x-x_0|}{R+Z} + \frac{2d^2-b^2}{b}\ln(x+R) + a\frac{x-2x_0}{x-x_0}R\right), \tag{23}$$

$$R=\sqrt{x^2+d^2}, \quad Z = ax+b. \tag{24}$$

Now it is seen that singular terms containing $\ln|x-x_0|$ cancels in $F_i^{(1)} - F_i^{(2)}(x)$, as expected. Also, for $x_0\in[x_{01},x_{02}]$, singularities $\dfrac{1}{x-x_0}$ are removable.

One way to do this is to transform these singular expressions into a form that is regular in $x=x_0$. Depending on the sign of b, these variants can be denoted by curl brackets for the two cases, $\left\{\begin{matrix} b>0 \\ b<0\end{matrix}\right\}$. Final solution is

$$F_i(x) = \left\{\begin{matrix} \left\{\begin{matrix} \dfrac{x+x_0}{Rb+d^2} \\ \dfrac{Rb-d^2}{b^2(x-x_0)} \end{matrix}\right\} \\ -\dfrac{ad^2}{b^2}\ln(x+R) + \dfrac{d^2}{b^2}\ln(R+Z) + x_0\left\{\begin{matrix} \dfrac{x+x_0}{Rb+d^2} \\ \dfrac{Rb-d^2}{b^2(x-x_0)} \end{matrix}\right\} \\ \dfrac{d^2}{b^2}\left(x+2x_0\ln(R+Z) + \dfrac{b^2-2d^2}{b}\ln(x+R)\right) \\ -\left\{\begin{matrix} \left(\dfrac{d^2}{b^2}-1\right)\dfrac{R^2(3x_0-x)-x_0^2(x+x_0)}{x_0^2-(x-2x_0)aR} \\ \dfrac{d^2}{b^2}\dfrac{x_0^2+(x-2x_0)aR}{x-x_0} \end{matrix}\right\} \end{matrix}\right\}, \tag{25}$$

where for $a=0$ ($x_0=0$), expressions in the lowest curl brackets should be explicitly equated to zero.

Another way to make the solution regular in $x=x_0$ is to use the limit of the function for $x\to x_0$. First, $x=x_0$ is possible only if $b>0$, because $Z(x_0)=\dfrac{d^2}{b}\geq 0$. Also, for $b>0$, $R(x_0)=\dfrac{d^2}{b}$. The first limit is

$$\lim_{x\to x_0}\frac{Rb-d^2}{b^2(x-x_0)} = \frac{a}{b} \tag{26}$$

and the second is

$$\lim_{x\to x_0}\left(\frac{d^2}{b^2}\,\frac{x_0{}^2+(x-2x_0)aR}{x-x_0}\right)=\frac{ad^2}{b}=x_0. \tag{27}$$

Using the curl brackets notation for the cases $\left\{\begin{array}{c} x=x_0\\ x\neq x_0\end{array}\right\}$, this form of the final solution is written as

$$F_i(x)=\left\{\begin{array}{c}\left\{\begin{array}{c}\dfrac{a}{b}\\ \dfrac{Rb-d^2}{b^2(x-x_0)}\end{array}\right\}\\ -\dfrac{ad^2}{b^2}\ln(x+R)+\dfrac{d^2}{b^2}\ln(R+Z)+x_0\left\{\begin{array}{c}\dfrac{a}{b}\\ \dfrac{Rb-d^2}{b^2(x-x_0)}\end{array}\right\}\\ \dfrac{d^2}{b^2}\left(x+2x_0\ln(R+Z)+\dfrac{b^2-2d^2}{b}\ln(x+R)\right)\\ -\left\{\begin{array}{c}\dfrac{ad^2}{b}\\ \dfrac{d^2}{b^2}\,\dfrac{x_0{}^2+(x-2x_0)aR}{x-x_0}\end{array}\right\}\end{array}\right\}. \tag{28}$$

In this form no special attention to the $a=0$ case is necessary.

After numerical implementation of these two methods of singularity removal, the first method (given by (25)) is found to be stable, while the second method (given by (28)) is found to be unstable in the vicinity of x_0.

6. GENERAL SOLUTION

6.1. First part of the solution

In this case no special relations between the integral parameters will be assumed. In other words, $d>0$, $a\neq\pm1$ and $L>0$. From this follows $A\neq0$ and from restriction 4 follows $b\neq\pm d$ and $C\neq0$. Function P^2 is given by (8) and (9),

$$F_i(x)=F_i^{(1)}-F_i^{(2)} \tag{29}$$

$$=\int\frac{x^i}{Ax^2+Bx+C}\,\mathrm{d}x-\int x^i\frac{ax+b}{\sqrt{x^2+d^2}\,(Ax^2+Bx+C)}\,\mathrm{d}x. \tag{30}$$

The first integral is a classical one. Change of variables $\dfrac{Ax+B/2}{L}=t$ transforms the function P^2 to the form

$$P^2=(1+t^2)\,\frac{L^2}{A}. \tag{31}$$

This integral now has the form

$$F_i^{(1)} = \frac{1}{LA^i} \int \frac{(Lt - B/2)^i}{1+t^2}\,\mathrm{d}t, \tag{32}$$

which is a combination of table integrals. The solution is

$$F_i^{(1)} = \left\{ \begin{array}{c} \dfrac{1}{L}\arctan\dfrac{Ax+B/2}{L} \\ \dfrac{1}{A}\ln P - \dfrac{B/2}{AL}\arctan\dfrac{Ax+B/2}{L} \\ \dfrac{B^2/2-AC}{A^2L}\arctan\dfrac{Ax+B/2}{L} - \dfrac{B}{A^2}\ln P + \dfrac{x}{A} \end{array} \right\}. \tag{33}$$

The solution of the second integral requires more systematic approach. Change of variables $x = d\tan t$ is first applied to eliminate the square root function. Next, another change of variables, $y = \tan\frac{t}{2}$, is applied to transform the integrand into a rational function. Overall change of variables is

$$x = 2d\,\frac{y}{1-y^2}, \tag{34}$$

a 'tangent-like' function that maps the interval $x \in (-\infty, +\infty)$ to $y \in]-1, +1[$. The inverse transformation is the solution of (34) for y,

$$y_{1,2} = \frac{-d \pm \sqrt{x^2+d^2}}{x} = \frac{-d \pm R}{x}. \tag{35}$$

However, for proper $x \leftrightarrow y$ mapping the '+' sign in (35) should be used. Moreover, to make the expression regular in $x = 0$ it should be rationalized, finally resulting in

$$y = \frac{x}{d+\sqrt{x^2+d^2}} = \frac{x}{d+R}. \tag{36}$$

After this change of variables the integral results in

$$F_i^{(2)} = d^i 2^{i+1} \int \frac{y^i(2ady + b - by^2)}{(1-y)^i(1+y)^i\big(4Ad^2y^2 + 2Bd(1-y^2)y + C(1-y^2)^2\big)}\,\mathrm{d}y. \tag{37}$$

A classical method for integration of rational functions is then applied. Main task in the solution is factorization of the polynomial in the denominator. This can be done in several ways. Factorization can be performed into simple complex factors of the $(y-y_j)$ form, or into real factors. Depending on type of the polynomial roots (real roots or complex conjugate pairs of complex roots), real factors are of the $(y-y_j)$ or $(y^2+\alpha y+\beta)$ form. We choose to perform real factorization. However, in both cases roots of polynomial have to be found.

Let us denote the polynomial in square brackets of (37) by $f(x)$,

$$f(x) = 4Ad^2y^2 + 2Bd(1-y^2)y + C(1-y^2)^2. \tag{38}$$

Observation that the roots of the original function, $P^2(x)$, are also roots of $f(x)$ directly solves for the roots of $f(x)$. Applying transformation (35) to roots of the P^2 function, we get

$$y_{1,3} = \frac{-d \pm R(x_1)}{x_1}, \quad y_{2,4} = \frac{-d \pm R(x_2)}{x_2} = y^*_{1,3}, \tag{39}$$

where numbering $1, 2, 3, 4$ is arbitrary, an asterisk denotes the complex conjugate and $R(x) = \sqrt{x^2 + d^2}$. To express $y_{1,2,3,4}$ by basic parameters, d, b and a, it is beneficial to notice that for $x = x_{1,2}$, $P = 0$ and (complex) $R(x_{1,2}) = \pm(ax_{1,2}+b) = \pm Z(x_{1,2})$. Equations (39) transform into

$$y_{1,3} = \frac{-d \pm (ax_1 + b)}{x_1}, \quad y_{2,4} = \frac{-d \pm (ax_2 + b)}{x_2}. \tag{40}$$

Substituting $x_{1,2}$ from (13) and rationalizing the denominator, after simple transformations we obtain

$$y_{1,2} = \frac{ad \pm jL}{d + b}, \quad y_{3,4} = \frac{-ad \pm jL}{d - b}. \tag{41}$$

6.2. Real factorization

Real quadratic factors of $f(x)$ are obtained by combining complex conjugate pairs of roots. Taking into account $C = d^2 - b^2 = (d - b)(d + b)$, we finally obtain $f(x)$ in the factorized form,

$$f(x) = M_1 \cdot M_2, \quad M_{1,2} = (d \pm b)y^2 \mp 2ady + (d \mp b). \tag{42}$$

Now the integrand in (37), which we now denote by $N(y)$, is decomposed as the rational function:
For $i = 0$,

$$N(y) = \frac{py + q}{M_1} + \frac{ry + s}{M_2}. \tag{43}$$

For $i = 1$,

$$N(y) = \frac{py + q}{M_1} + \frac{ry + s}{M_2} + \frac{u}{1 - y} + \frac{v}{1 + y}. \tag{44}$$

For $i = 2$,

$$N(y) = \frac{py + q}{M_1} + \frac{ry + s}{M_2} + \frac{u}{1 - y} + \frac{v}{1 + y} + \frac{w}{(1 - y)^2} + \frac{t}{(1 + y)^2}. \tag{45}$$

Solving for unknown factors p, q, r, s, u, v, w and t is straightforward. This decomposes the integral (37) into a set of table integrals, resulting in

$$F_i^{(2)}(x) = \left\{ \begin{array}{l} \frac{1}{L}\left(\operatorname{arctg}\frac{(b-d)y-ad}{L} + \operatorname{arctg}\frac{(b+d)y-ad}{L}\right) \\ \frac{1}{A}\left\{\frac{ab}{L}\left(\operatorname{arctg}\frac{(b-d)y-ad}{L} + \operatorname{arctg}\frac{(b+d)y-ad}{L}\right)\right. \\ \qquad \left. +a\ln\frac{1+y}{1-y} + \frac{1}{2}\ln\frac{(d+b)^2-2ady+(d-b)}{(d-b)y^2+2ady+(d+b)}\right\} \\ \frac{1}{A}\left\{\left(\operatorname{arctg}\frac{(b-d)y-ad}{L} + \operatorname{arctg}\frac{(b+d)y-ad}{L}\right)\times\right. \\ \qquad \times\frac{b^2(1+a^2)-d^2(1-a^2)}{(1-a^2)L} + b\frac{1+a^2}{1-a^2}\ln\frac{1+Y}{1-y} \\ \qquad \left. +\frac{ab}{1-a^2}\ln\frac{(d+b)^2-2ady+(d-b)}{(d-b)y^2+2ady+(d+b)} + \frac{2ad}{1-y^2}\right\} \end{array} \right\}, \tag{46}$$

$$y = \frac{x}{d+R}. \tag{47}$$

Complete solution is given by combining (33) and (46) in $F_i(x) = F_i^{(1)} - F_i^{(2)}$.

Note: Solution for $F_0^{(2)}$ can also be obtained in the form

$$F_0^{(2)} = \left\{ \begin{array}{ll} \frac{\pi}{2}\operatorname{sgn}\big(b(x-x_0)\big) & \\ \qquad + \arctan\frac{x^2(b^2-L^2)-2abd^2x+d^2(a^2d^2-L^2)}{2LRb(x-x_0)}, & x \neq x_0 \\ 0, & x = x_0 \end{array} \right\}, \tag{48}$$

$$x_0 = \frac{ad^2}{b}. \tag{49}$$

However, this needs analysis of the $b = 0$ as a special case.

6.3. Complex factorization

Factorization of $f(x)$ into simple complex factors is

$$f(x) = C(y-y_1)(y-y_2)(y-y_3)(y-y_4). \tag{50}$$

This results in indefinite integral components of the form $\operatorname{Re}\{\log(y-y_i)\}$.

From (41) and $L \neq 0$ follows that $\operatorname{Im}\{y_{1,2,3,4}\} \neq 0$. Also, y is real on the integration path. This has two important consequences, both beneficial for the solution. The first is that the above logarithm functions are always defined ($(y-y_i) \neq 0$). The second is that the trajectory of these log functions in the complex plane do not cut the branch cut (negative real axis). Thus, these functions are regular in the domain of the definite integral limits.

Solution for $F_i^{(2)}(x)$ is now obtained similarly to a real factorization. The result is

$$
(51)\quad F_i^{(2)}(x) = \left\{ \begin{array}{l} \dfrac{4}{C}\,\mathrm{Re}\left\{ \dfrac{x_2(ax_1+b)}{(x_1-x_2)(y_1-y_3)}\big(\ln(y-y_1)-\ln(y-y_3)\big)\right\} \\ \dfrac{1}{A}\left[a\ln\dfrac{1+y}{1-y} \right. \\ \qquad \left. +4\,\mathrm{Re}\left\{ \dfrac{ax_1+b}{(x_1-x_2)(y_1-y_3)}\big(\ln(y-y_1)-\ln(y-y_3)\big)\right\}\right] \\ \dfrac{1}{A}\left[b\,\dfrac{1+a^2}{1-a^2}\ln\dfrac{1+y}{1-y}+\dfrac{2ad}{1-y^2} \right. \\ \qquad \left. +4\,\mathrm{Re}\left\{ \dfrac{x_1(ax_1+b)}{(x_1-x_2)(y_1-y_3)}\big(\ln(y-y_1)-\ln(y-y_3)\big)\right\}\right] \end{array} \right\},
$$

where $x_{1,2}$ are given by (13), and $y_{1,3}$ by (41).

This concludes the complete solution of the integral.

REFERENCES

1. B. D. Popović, M. B. Dragović, A. R. Đorđević: *Analysis and Synthesis of Wire Antennas and Scatterers.* Chichester UK: Research Studies Press (J Wiley); 1982.
2. A. R. Đorđević, M. B. Baždar, V. V. Petrović, D. I. Olćan, T. K. Sarkar, R. F. Harrington: Awas for Windows, ver. 2.0, Analysis of Wire Antennas and Scatterers, User's Manual. Boston MA: Artech House; 2002.
3. A. Sommerfeld: *Über die Ausbreitung der Wellen in der Drachtlosen Telegraphie.* Ann. Phys. Lpz., **28** (1909), 665–736.
4. N. Hojjat, S. Safavi-Naeini, R. Faraji-Dana, Y. L. Chow: *Fast computation of the nonsymmetric components of the Green's function for multilayer media using complex images.* IEE Proc-Microw. Antennas Propag., **145** (1998), 285–288.
5. E. Simsek, O. H. Liu, B. Wei: Singularity subtraction for evaluation of Green's functions for multilayer media. IEEE Trans. Microwave Theory Tech., **54** (2006), 216–225.
6. V. V. Petrović, A. R. Đorđević: *General singularity extraction technique for reflected Sommerfeld integrals.* Arch. Elektr. Übertr. (AEUE) (in press).

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