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AN EXTENSION OF SOME INTEGRAL INEQUALITIES

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Inspired by a result given in the paper Bai-Ni Guo, Xin Jiang: *Some integral inequalities.* Publ. Elektrotehn. Fak. Ser. Math., **10** (1999), 27–29, we prove an integral inequality that involves several functions.

We denote by $\mathbb{R}^n$, $n \geq 1$, the Euclidean space of dimension n endowed by the standard Lebesgue measure dx. Let $L^1(\Omega)$ be the space of real functions defined in $\Omega \subset \mathbb{R}^n$ such that $\int_\Omega f(x)\,\mathrm{d}x < \infty$.

In their paper [**1**], Bai-Ni Guo and Xin Jiang state the following:

Theorem 1. *Let Ω be a domain in $\mathbb{R}^n$ and $f, g \in L^1(\Omega)$ such that $f \geq 0$ and $g \geq 0$ and let $I(f) = \int_\Omega f(x)\,\mathrm{d}x$. Further let $h : \Omega \to \mathbb{R}$ such that $h^2 \in L^1(\Omega)$. If*

$$I(f)I(gh) = I(fh)I(g), \tag{1}$$

then

$$(fh^2)\big(I(g)\big)^2 + I(gh^2)\big(I(f)\big)^2 \geq I(fh)I(gh)I(f+g) \tag{2}$$

and the equality case is valid if and only if h is constant.

There is a misprint in the formulation of the above theorem: instead of assumption $f \geq 0$ and $g \geq 0$, it should read $I(f) > 0$ and $I(g) > 0$. Indeed if $\Omega = \left[-\dfrac{\pi}{2}, \pi\right]$, $f : x \mapsto \sin(x)$, $g : x \mapsto \cos(x)$ and

$$h : x \mapsto \begin{cases} 1 & \text{if } x \in \left[-\dfrac{\pi}{2}, -\dfrac{\pi}{4}\right] \cup \left[\dfrac{3\pi}{4}, \pi\right] \\ 0 & \text{otherwise} \end{cases}, \text{ (notice that } h^2 = h),$$

then we have $\int_\Omega f = \int_\Omega g = 1$, $\int_\Omega fh = \int_\Omega gh = 1 - \sqrt{2}$, while (2) is false.

We will give another formulation of the Theorem 1, which permits us to extend it to the case of several functions.

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Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ and $f, g \in L^1(\Omega)$ such that $f(x) \geq 0$ and $g(x) \geq 0$ for all $x \in \Omega$. Further let $h : \Omega \to \mathbb{R}$ such that $h^2 \in L^1(\Omega)$. If $I(f)I(gh) = I(fh)I(g)$, then*

$$\begin{vmatrix} I(g) & I(fh) \\ I(gh) & I(fh^2) \end{vmatrix} \geq 0 \quad \text{and} \quad \begin{vmatrix} I(f) & I(gh) \\ I(fh) & I(gh^2) \end{vmatrix} \geq 0, \tag{3}$$

$$I(f) \begin{vmatrix} I(f) & I(gh) \\ I(fh) & I(gh^2) \end{vmatrix} + I(g) \begin{vmatrix} I(g) & I(fh) \\ I(gh) & I(fh^2) \end{vmatrix} \geq 0. \tag{4}$$

Proof. By SCHWARZ inequality

$$\big(I(fh)\big)^2 \leq I(fh^2)I(f),$$

we have

$$\begin{vmatrix} I(f) & I(fh) \\ I(fh) & I(fh^2) \end{vmatrix} \geq 0 \quad \text{and then} \quad \begin{vmatrix} I(f)I(g) & I(fh) \\ I(fh)I(g) & I(fh^2) \end{vmatrix} \geq 0.$$

Using the hypothesis $I(f)I(gh) = I(fh)I(g)$, we obtain

$$I(f) \begin{vmatrix} I(g) & I(fh) \\ I(gh) & I(fh^2) \end{vmatrix} \geq 0,$$

and the inequality follows for $I(f) \neq 0$ (the case $I(f) = 0$ is trivial, it implies that $I(fh) = 0$).

Using the same reasoning process, we obtain the dual case:

$$\begin{vmatrix} I(f) & I(gh) \\ I(fh) & I(gh^2) \end{vmatrix} \geq 0.$$

The last inequality is a trivial combination of the first and the second inequalities.

□

REMARK. If we assume that $I(f)I(g) \neq 0$, the equality holds if and only if h is constant.

The formulation of the Theorem 2 suggests the following extension. We will start with the case of three functions, it permits us to understand the evolution of the hypothesis and the results.

Theorem 3. (The case of three functions) *Let $\Omega \subset \mathbb{R}^n$ and $f, g, h \in L^1(\Omega)$ such that $f(x) \geq 0$, $g(x) \geq 0$ and $h(x) \geq 0$, for all $x \in \Omega$. Further let $u : \Omega \to \mathbb{R}$ such that $u^k \in L^1(\Omega)$, for $k = 1, 2, 3, 4$. We denote*

$$\Delta(f, g, h) = \begin{vmatrix} I(f) & I(gu) & I(hu^2) \\ I(fu) & I(gu^2) & I(hu^3) \\ I(fu^2) & I(gu^3) & I(hu^4) \end{vmatrix}.$$

If for $r = 1, 2, 3$, we have

$$\begin{aligned} I(f)I(gu^r) &= I(fu^r)I(g), \\ I(g)I(hu^r) &= I(gu^r)I(h), \\ I(h)I(fu^r) &= I(hu^r)I(f), \end{aligned}$$

then

$$\Delta(f, g, h) \geq 0, \ \Delta(g, h, f) \geq 0, \ \Delta(h, f, g) \geq 0, \ \text{and}$$
$$\big(I(f)\big)^2 \Delta(f, g, h) + \big(I(g)\big)^2 \Delta(g, h, f) + \big(I(h)\big)^2 \Delta(h, f, g) \geq 0.$$

If we assume that $I(f)I(g)I(h) \neq 0$, the equality holds if and only if u is constant.

Theorem 4. (The general case) *Let $\Omega \subset \mathbb{R}^n$ and $f_1, f_2, \dots, f_m \in L^1(\Omega)$, $m \geq 2$ such that $f_k(x) \geq 0$ for all $x \in \Omega$ and for $k = 1, \dots, m$.*

Further let $u : \Omega \to \mathbb{R}$ such that $u^k \in L^1(\Omega)$ for $k = 1, \dots, 2m-2$.

We denote for $\sigma \in \mathcal{S}_m$ (group of permutations of $\{1, \dots, m\},$)

$$\Delta_\sigma(f_1, f_2, \dots, f_m) = \begin{vmatrix} I(f_{\sigma(1)}) & I(f_{\sigma(2)}u) & \cdots & I(f_{\sigma(m)}u^{m-1}) \\ I(f_{\sigma(1)}u) & I(f_{\sigma(2)}u^2) & \cdots & I(f_{\sigma(m)}u^m) \\ \vdots & \vdots & \ddots & \vdots \\ I(f_{\sigma(1)}u^{m-1}) & I(f_{\sigma(2)}u^m) & \cdots & I(f_{\sigma(m)}u^{2m-2}) \end{vmatrix}.$$

If for $r = 1, \dots, 2m-3$; and for $i, j : 1 \leq i \neq j \leq m$, we have

$$I(f_i)I(f_j u^r) = I(f_i u^r)I(f_j)$$

then

$$\Delta_\sigma(f_1, f_2, \dots, f_m) \geq 0 \quad \textit{for all } \sigma \in \mathcal{S}_m \textit{ with } \sigma \textit{ be a circular permutation,} \tag{5}$$

$$\sum_{\sigma \in \mathcal{S}_m/\sigma \ \textit{circular}} \left(I(f_{\sigma(1)})\right)^{m-1} \Delta_\sigma(f_1, f_2, \dots, f_m) \geq 0. \tag{6}$$

If we assume that $\prod_{k=1}^{m} I(f_k) \neq 0$, the equality holds if and only if u is constant.

Proof (of the Theorem 4). It easy to see that (6) is a consequence of (5), let us prove the relation (5).

Without lose the generality, we do the proof for $\sigma = Id$.

Denoting $G(x_1, x_2, \dots, x_m) = \det\left(\langle x_i, x_j\rangle\right)_{i,j=1,\dots,m}$, the well known Gram's inequality gives $G(x_1, x_2, \dots, x_m) \geq 0$. Further, we have

$$G\left(f_m^{1/2}, f_m^{1/2}u, \dots, f_m^{1/2}u^{m-1}\right) \geq 0, \text{ with } \langle f, g\rangle = \int_\Omega f(x)g(x)\,\mathrm{d}x.$$

Then

$$\begin{aligned} & I(f_1)I(f_2)\cdots I(f_{m-1})G(f_m^{1/2}, f_m^{1/2}u, \dots, f_m^{1/2}u^{m-1}) \\ &= \prod_{j=1}^{m-1} I(f_j) \times \begin{vmatrix} I(f_m) & I(f_m u) & \cdots & I\left(f_m u^{m-2}\right) & I\left(f_m u^{m-1}\right) \\ I(f_m u) & I\left(f_m u^2\right) & \cdots & I\left(f_m u^{m-1}\right) & I\left(f_m u^m\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I\left(f_m u^{m-1}\right) & I(f_m u^m) & \cdots & I\left(f_m u^{2m-3}\right) & I\left(f_m u^{2m-2}\right) \end{vmatrix} \\ &= \begin{vmatrix} I(f_1)\, I(f_m) & \cdots & I(f_{m-1})\, I\left(f_m u^{m-2}\right) & I\left(f_m u^{m-1}\right) \\ I(f_1)\, I(f_m u) & \cdots & I(f_{m-1})\, I\left(f_m u^{m-1}\right) & I\left(f_m u^m\right) \\ \vdots & \ddots & \vdots & \vdots \\ I(f_1)\, I\left(f_m u^{m-1}\right) & \cdots & I(f_{m-1})\, I\left(f_m u^{2m-3}\right) & I\left(f_m u^{2m-2}\right) \end{vmatrix} \geq 0 \end{aligned}$$

and by using the hypothesis, we obtain

$$\begin{vmatrix} I(f_1)\, I(f_m) & I(f_2u)\, I(f_m) & \cdots & I\left(f_{m-1}u^{m-2}\right) I(f_m) & I\left(f_m u^{m-1}\right) \\ I(f_1u)\, I(f_m) & I\left(f_2u^2\right) I(f_m) & \cdots & I\left(f_{m-1}u^{m-1}\right) I(f_m) & I\left(f_m u^{m}\right) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ I\left(f_1u^{m-1}\right) I(f_m) & I\left(f_2u^{m}\right) I(f_m) & \cdots & I\left(f_{m-1}u^{2m-3}\right) I(f_m) & I\left(f_m u^{2m-2}\right) \end{vmatrix} \geq 0$$

which is equivalent to

$$\big(I(f_m)\big)^{m-1}\Delta_{Id}(f_1, f_2, \ldots, f_m) \geq 0,$$

i.e.

$$\Delta_{Id}(f_1, f_2, \ldots, f_m) \geq 0 \quad \text{when } I(f_m) \neq 0.$$

The case $I(f_m) = 0$ is trivial ($\Rightarrow \int_\Omega f_m u = 0 \Rightarrow \int_\Omega f_m u.u = 0 \quad \cdots$). □

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