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SOME SHARP OSTROWSKI-GRÜSS TYPE INEQUALITIES

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Using a variant of Grüss inequality, to give a new proof of a well known result on Ostrowski-Grüss type inequalities and sharpness of this inequality is obtained. Moreover, a new general sharp Ostrowski-Grüss type inequality is given.

1. INTRODUCTION

In 2001, Cheng in [**3**] has improved and further generalized some Ostrowski-Grüss type inequalities involving bounded once and twice differentiable mappings.

In 2002, almost at the same time, Cheng and Sun in [**4**] as well as Matić in [**5**] have established the following variant of Grüss inequality.

Lemma 1. *Let $h, g : [a,b] \to \mathbb{R}$ be two integrable functions such that $\gamma \le g(t) \le \Gamma$ for all $t \in [a,b]$, where $\gamma, \Gamma \in \mathbb{R}$ are constants. Then*

$$(1)\quad \left|\int_a^b h(t)g(t)\,\mathrm{d}t - \frac{1}{b-a}\int_a^b h(t)\,\mathrm{d}t\int_a^b g(t)\,\mathrm{d}t\right| \le \frac{\Gamma-\gamma}{2}\int_a^b \left|h(t) - \frac{1}{b-a}\int_a^b h(y)\,\mathrm{d}y\right|\mathrm{d}t.$$

Moreover, Matić has proved that there exists function g to attain the equality in (1), Cerone and Dragomir have proved in [**3**] that $1/2$ in (1) is sharp constant.

In Theorem 3 of [**2**], Cerone and Dragomir have treated Theorem 1.5 of [**3**] in a more general setup by using Lemma 1 and obtain

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a function which is absolutely continuous on $[a,b]$ and there exist constants $\gamma_1, \Gamma_1 \in \mathbb{R}$ such that $\gamma_1 \le f'(t) \le \Gamma_1$ for a.e.$t \in [a,b]$. Then for all $x \in [a,b]$, we have*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{f(b)-f(a)}{b-a}\left(x - \frac{a+b}{2}\right)\right| \le \frac{1}{8}(b-a)(\Gamma_1-\gamma_1),$$

where the constant $1/8$ is sharp.

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In this paper, we will also treat Theorem 1.6, Theorem 3.1 and Theorem 3.2 of [**3**] by using Lemma 1 to obtain some sharp OSTROWSKI-GRÜSS type inequalities as follows:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be such that f' is absolutely continuous on $[a,b]$ and there exist constants $\gamma_2, \Gamma_2 \in \mathbb{R}$ such that $\gamma_2 \le f''(t) \le \Gamma_2$ for a.e.$t \in [a,b]$. Then for all $x \in [a,b]$, we have*

$$\begin{aligned}(2)\qquad &\Bigg| f(x) - \Big(x - \frac{a+b}{2}\Big) f'(x) + \Big(\frac{1}{24}(b-a)^2 + \frac{1}{2}\Big(x - \frac{a+b}{2}\Big)^2\Big) \frac{f'(b)-f'(a)}{b-a} \\ &\qquad - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \Bigg| \le (\Gamma_2 - \gamma_2)\, G(a,b,x),\end{aligned}$$

where

$$(3)\qquad G(a,b,x)$$

$$= \begin{cases} \dfrac{1}{3(b-a)}\left(\left|(x-a)\Big(x - \dfrac{a+b}{2}\Big)(b-x)\right| \right. \\ \qquad \left. + \Big(\dfrac{1}{12}(b-a)^2 + \Big(x - \dfrac{a+b}{2}\Big)^2\Big)^{3/2}\right), & a \le x \le \dfrac{1}{3}(2a+b), \\ & \dfrac{1}{3}(a+2b) \le x \le b, \\ \dfrac{2}{3(b-a)}\Big(\dfrac{1}{12}(b-a)^2 + \Big(x - \dfrac{a+b}{2}\Big)^2\Big)^{3/2}, & \dfrac{1}{3}(2a+b) \le x \le \dfrac{1}{3}(a+2b). \end{cases}$$

The inequality (2) *with* (3) *is sharp.*

Theorem 3. *Let the assumptions of Theorem* 1 *hold. Then for all $x \in [a,b]$, we have*

$$\begin{aligned}(4)\qquad &\left| \frac{1}{2} f(x) - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{(x-b)f(b) - (x-a)f(a)}{2(b-a)} \right| \\ &\qquad \le \frac{1}{8(b-a)}\big((x-a)^2 + (x-b)^2\big)(\Gamma_1 - \gamma_1).\end{aligned}$$

The constant $1/8$ is sharp.

Theorem 4. *Let the assumptions of Theorem* 2 *hold. Then for all $x \in [a,b]$, we have*

$$\begin{aligned}(5)\qquad &\left| f(x) - \frac{2}{3}\Big(x - \frac{a+b}{2}\Big) f'(x) + \frac{(x-b)^2 f'(b) - (x-a)^2 f'(a)}{6(b-a)} - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t \right| \\ &\qquad \le \frac{1}{9\sqrt{3}(b-a)}\big((x-a)^3 + (b-x)^3\big)(\Gamma_2 - \gamma_2).\end{aligned}$$

The constant $\dfrac{1}{9\sqrt{3}}$ is sharp.

Here we have given revised version for (5) since the expression in [**3**] contained a misprint.

In Section 2, we will use Lemma 1 to provide a new proof of Theorem 2. Instead of proving Theorem 3 and Theorem 4, in Section 3, we will give a new general sharp OSTROWSKY-GRÜSS type inequality.

2. A NEW PROOF OF THEOREM 2

We choose in (1), $h(t) = K_2(x,t)$ and $g(t) = f''(t)$, where $K_2 : [a,b]^2 \to \mathbb{R}$ is given by

$$K_2(x,t) := \begin{cases} \dfrac{(t-a)^2}{2}, & a \le t < x, \\ \dfrac{(t-b)^2}{2}, & x \le t \le b. \end{cases}$$

Then we have

$$\int_a^b K_2(x,t)\,\mathrm{d}t = \frac{(x-a)^3-(x-b)^3}{6} = \left(\frac{1}{24}(b-a)^2 + \frac{1}{2}\Big(x-\frac{a+b}{2}\Big)^2\right)(b-a),$$

and so

$$\int_a^b \left|h(t) - \frac{1}{b-a}\int_a^b h(y)\,\mathrm{d}y\right|\mathrm{d}t = \int_a^x \left|\frac{(t-a)^2}{2} - \left(\frac{1}{24}(b-a)^2 + \frac{1}{2}\Big(x-\frac{a+b}{2}\Big)^2\right)\right|\mathrm{d}t + \int_x^b \left|\frac{(t-b)^2}{2} - \left(\frac{1}{24}(b-a)^2 + \frac{1}{2}\Big(x-\frac{a+b}{2}\Big)^2\right)\right|\mathrm{d}t.$$

Denote $t_1 = a + \left(\frac{1}{12}(b-a)^2 + \Big(x-\frac{a+b}{2}\Big)^2\right)^{1/2}$ and $t_2 = b - \left(\frac{1}{12}(b-a)^2 + \Big(x-\frac{a+b}{2}\Big)^2\right)^{1/2}$. It is clear that $a < t_1 < t_2 < b$.

In case $a \le x \le \frac{2a+b}{3}$, we see that $a \le x \le t_1$, and hence

$$\begin{aligned}\int_a^b \left|h(t) - \frac{1}{b-a}\int_a^b h(y)\,\mathrm{d}y\right|\mathrm{d}t &= \int_a^x \left(\frac{1}{24}(b-a)^2 + \frac{1}{2}\Big(x-\frac{a+b}{2}\Big)^2 - \frac{(t-a)^2}{2}\right)\mathrm{d}t \\ &+ \int_x^{t_2} \left(\frac{(t-b)^2}{2} - \frac{1}{24}(b-a)^2 - \frac{1}{2}\Big(x-\frac{a+b}{2}\Big)^2\right)\mathrm{d}t \\ &+ \int_{t_2}^b \left(\frac{1}{24}(b-a)^2 + \frac{1}{2}\Big(x-\frac{a+b}{2}\Big)^2 - \frac{(t-b)^2}{2}\right)\mathrm{d}t \\ &= \frac{2}{3}\left((x-a)\Big(\frac{a+b}{2}-x\Big)(b-x) + \left(\frac{1}{12}(b-a)^2 + \Big(x-\frac{a+b}{2}\Big)^2\right)^{3/2}\right).\end{aligned}$$

In case $\frac{2a+b}{3} \le x \le \frac{a+2b}{3}$, we see that $t_1 \le x \le t_2$, and hence

$$\begin{aligned}
\int_a^b \left|h(t) - \frac{1}{b-a}\int_a^b h(y)\,\mathrm{d}y\right|\mathrm{d}t &= \int_a^{t_1} \left(\frac{1}{24}(b-a)^2 + \frac{1}{2}\Big(x - \frac{a+b}{2}\Big)^2 - \frac{(t-a)^2}{2}\right)\mathrm{d}t \\
&+ \int_{t_1}^{x} \left(\frac{(t-a)^2}{2} - \frac{1}{24}(b-a)^2 - \frac{1}{2}\Big(x - \frac{a+b}{2}\Big)^2\right)\mathrm{d}t \\
&+ \int_{x}^{t_2} \left(\frac{(t-b)^2}{2} - \frac{1}{24}(b-a)^2 - \frac{1}{2}\Big(x - \frac{a+b}{2}\Big)^2\right)\mathrm{d}t \\
&+ \int_{t_2}^{b} \left(\frac{1}{24}(b-a)^2 + \frac{1}{2}\Big(x - \frac{a+b}{2}\Big)^2 - \frac{(t-b)^2}{2}\right)\mathrm{d}t \\
&= \frac{4}{3}\left(\frac{1}{12}(b-a)^2 + \Big(x - \frac{a+b}{2}\Big)^2\right)^{3/2}.
\end{aligned}$$

In case $\frac{a+2b}{3} \le x \le b$, we see that $t_2 \le x \le b$, and hence

$$\begin{aligned}
\int_a^b \left|h(t) - \frac{1}{b-a}\int_a^b h(y)\,\mathrm{d}y\right|\mathrm{d}t &= \int_a^{t_1} \left(\frac{1}{24}(b-a)^2 + \frac{1}{2}\Big(x - \frac{a+b}{2}\Big)^2 - \frac{(t-a)^2}{2}\right)\mathrm{d}t \\
&+ \int_{t_1}^{x} \left(\frac{(t-a)^2}{2} - \frac{1}{24}(b-a)^2 - \frac{1}{2}\Big(x - \frac{a+b}{2}\Big)^2\right)\mathrm{d}t \\
&+ \int_{x}^{b} \left(\frac{1}{24}(b-a)^2 + \frac{1}{2}\Big(x - \frac{a+b}{2}\Big)^2 - \frac{(t-b)^2}{2}\right)\mathrm{d}t \\
&= \frac{2}{3}\left((x-a)\Big(x - \frac{a+b}{2}\Big)(b-x) + \left(\frac{1}{12}(b-a)^2 + \Big(x - \frac{a+b}{2}\Big)^2\right)^{3/2}\right).
\end{aligned}$$

Thus by Lemma 1, we can derive

$$\begin{aligned}
&\left|f(x) - \Big(x - \frac{a+b}{2}\Big)f'(x) + \left(\frac{1}{24}(b-a)^2 + \frac{1}{2}\Big(x - \frac{a+b}{2}\Big)^2\right)\frac{f'(b)-f'(a)}{b-a}\right. \\
&\qquad\qquad\left. - \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\right| \\
&= \left|\frac{1}{b-a}\int_a^b K_2(x,t)f''(t)\,\mathrm{d}t - \frac{1}{(b-a)^2}\int_a^b K_2(x,t)\,\mathrm{d}t\int_a^b f''(t)\,\mathrm{d}t\right|
\end{aligned}$$

$$
\leq \begin{cases} \dfrac{\Gamma_2-\gamma_2}{3\,(b-a)}\Bigg((x-a)\Big(\dfrac{a+b}{2}-x\Big)(b-x) \\ \qquad + \Big(\dfrac{1}{12}\,(b-a)^2+\Big(x-\dfrac{a+b}{2}\Big)^2\Big)^{3/2}\Bigg), & a \leq x \leq \dfrac{2a+b}{3}\,, \\ \dfrac{2\,(\Gamma_2-\gamma_2)}{3\,(b-a)}\Bigg(\dfrac{1}{12}\,(b-a)^2+\Big(x-\dfrac{a+b}{2}\Big)^2\Bigg)^{3/2}, & \dfrac{2a+b}{3} \leq x \leq \dfrac{a+2b}{3}\,, \\ \dfrac{\Gamma_2-\gamma_2}{3\,(b-a)}\Bigg((x-a)\Big(x-\dfrac{a+b}{2}\Big)(b-x) \\ \qquad + \Big(\dfrac{1}{12}\,(b-a)^2+\Big(x-\dfrac{a+b}{2}\Big)^2\Big)^{3/2}\Bigg), & \dfrac{a+2b}{3} \leq x \leq b, \end{cases}
$$

i.e., we have obtained the inequality (2) with (3).

It is not difficult to find that the inequality (2) with (3) is sharp. Indeed, we can construct the function $f(t)=\int_a^t\Big(\int_a^y j(z)\,\mathrm{d}z\Big)\mathrm{d}y$ to attain the equality in (2), where

$$
j(t)=\begin{cases} \gamma_2, & a \leq t < x, \\ \Gamma_2, & x \leq t < t_2, \\ \gamma_2, & t_2 \leq t \leq b, \end{cases} \qquad a \leq x \leq \frac{2a+b}{3}\,,
$$

$$
j(t)=\begin{cases} \gamma_2, & a \leq t < t_1, \\ \Gamma_2, & t_1 \leq t < t_2, \\ \gamma_2, & t_2 \leq t \leq b, \end{cases} \qquad \frac{2a+b}{3} \leq x \leq \frac{a+2b}{3}\,,
$$

$$
j(t)=\begin{cases} \gamma_2, & a \leq t < t_1, \\ \Gamma_2, & t_1 \leq t < x, \\ \gamma_2, & x \leq t \leq b, \end{cases} \qquad \frac{a+2b}{3} \leq x \leq b.
$$

The proof of Theorem 2 is complete.

3. A NEW GENERAL OSTROWSKY-GRÜSS TYPE INEQUALITY

We need the following two integral identities:

Lemma 2 [**1**]. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n \geq 1$. Then for all $x\in[a,b]$, we have the identity*:

$$
\int_a^b f(t)\,\mathrm{d}t=\sum_{k=0}^{n-1}\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!}\,f^{(k)}(x)+(-1)^n\int_a^b K_n(x,t)f^{(n)}(t)\,\mathrm{d}t,
$$

where the kernel $K_n:[a,b]^2\to\mathbb{R}$ is given by

$$
K_n(x,t):=\begin{cases} \dfrac{(t-a)^n}{n!}\,, & a \leq t < x, \\ \dfrac{(t-b)^2}{n!}\,, & x \leq t \leq b. \end{cases}
$$

Lemma 3. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n \geq 1$. Then for all $x \in [a,b]$, we have the identity:*

$$\int_a^b f(t)\,\mathrm{d}t = \sum_{k=0}^{n-1} \frac{(b-x)^{k+1} + (-1)^k (x-a)^{k+1}}{(k+1)!} f^{(k)}(x) \tag{6}$$
$$- \frac{(b-x)^n + (-1)^{n-1}(x-a)^n}{(n+1)!} f^{(n-1)}(x)$$
$$+ \frac{(b-x)^n f^{(n-1)}(b) + (-1)^{n-1}(x-a)^n f^{(n-1)}(a)}{(n+1)!}$$
$$+ (-1)^n \int_a^b H_n(x,t) f^{(n)}(t)\,\mathrm{d}t,$$

where the kernel $H_n : [a,b]^2 \to \mathbb{R}$ is given by

$$H_n(x,t) := \begin{cases} \dfrac{(t-a)^n}{n!} - \dfrac{(x-a)^n}{(n+1)!}, & a \leq t < x, \\[2mm] \dfrac{(t-b)^n}{n!} - \dfrac{(x-b)^n}{(n+1)!}, & x \leq t \leq b. \end{cases}$$

Proof. It is immediate that

$$\int_a^b H_n(x,t) f^{(n)}(t)\,\mathrm{d}t = \int_a^b K_n(x,t) f^{(n)}(t)\,\mathrm{d}t$$
$$+ \frac{(-1)^n (b-x)^n - (x-a)^n}{(n+1)!} f^{(n-1)}(x)$$
$$- \frac{(-1)^n (b-x)^n f^{(n-1)}(b) - (x-a)^n f^{(n-1)}(a)}{(n+1)!}.$$

Consequently, (6) follows from Lemma 2.

Now let us observe that

$$\int_a^b H_n(x,t)\,\mathrm{d}t = \int_a^x \left(\frac{(t-a)^n}{n!} - \frac{(x-a)^n}{(n+1)!} \right) \mathrm{d}t + \int_x^b \left(\frac{(t-b)^n}{n!} - \frac{(x-b)^n}{(n+1)!} \right) \mathrm{d}t = 0.$$

Further, denote $t_1 = a + \frac{1}{\sqrt[n]{n+1}}(x-a)$ and $t_2 = b - \frac{1}{\sqrt[n]{n+1}}(b-x)$. Clearly, $a < t_1 < t_2 < b$. If n is odd, we get

$$\int_a^b |H_n(x,t)|\,\mathrm{d}t = \int_a^{t_1} \left(\frac{(x-a)^n}{(n+1)!} - \frac{(t-a)^n}{n!} \right) \mathrm{d}t + \int_{t_1}^x \left(\frac{(t-a)^n}{n!} - \frac{(x-a)^n}{(n+1)!} \right) \mathrm{d}t$$
$$+ \int_x^{t_2} \left(\frac{(x-b)^n}{(n+1)!} - \frac{(t-b)^n}{n!} \right) \mathrm{d}t + \int_{t_2}^b \left(\frac{(t-b)^n}{n!} - \frac{(x-b)^n}{(n+1)!} \right) \mathrm{d}t$$
$$= \frac{2n}{(n+1)(n+1)!\,\sqrt[n]{n+1}} \left((x-a)^{n+1} + (b-x)^{n+1} \right)$$

and if n is even, we get

$$\begin{aligned}\int_a^b |H_n(x,t)|\,\mathrm{d}t &= \int_a^{t_1}\Big(\frac{(x-a)^n}{(n+1)!}-\frac{(t-a)^n}{n!}\Big)\mathrm{d}t+\int_{t_1}^{x}\Big(\frac{(t-a)^n}{n!}-\frac{(x-a)^n}{(n+1)!}\Big)\mathrm{d}t\\ &\quad+\int_x^{t_2}\Big(\frac{(t-b)^n}{n!}-\frac{(x-b)^n}{(n+1)!}\Big)\mathrm{d}t+\int_{t_2}^{b}\Big(\frac{(x-b)^n}{(n+1)!}-\frac{(t-b)^n}{n!}\Big)\mathrm{d}t\\ &=\frac{2n}{(n+1)(n+1)!\sqrt[n]{n+1}}\big((x-a)^{n+1}+(b-x)^{n+1}\big)\end{aligned}$$

Thus by Lemma 1 and Lemma 2 we can obtain a general Ostrowsky-Grüss type inequality as follows:

Theorem 5. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ for some $n\ge 1$ and there exist constants $\gamma_n,\Gamma_n\in\mathbb{R}$ such that $\gamma_n\le f^{(n)}(t)\le\Gamma_n$ for a.e.$t\in[a,b]$. Then for all $x\in[a,b]$, we have*

$$\begin{aligned}(7)\qquad &\bigg| f(x)-\frac{(b-x)^n+(-1)^{n-1}(x-a)^n}{(n+1)!(b-a)}\,f^{(n-1)}(x)\\ &+\sum_{k=1}^{n-1}\frac{(b-x)^{k+1}+(-1)^k(x-a)^{k+1}}{(k+1)!(b-a)}\,f^{(k)}(x)\\ &+\frac{(b-x)^nf^{(n-1)}(b)+(-1)^{n-1}(x-a)^nf^{(n-1)}(a)}{(n+1)!(b-a)}-\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t\bigg|\\ &\le\frac{n}{(n+1)(n+1)!\ \sqrt[n]{n+1}}\big((x-a)^{n+1}+(b-x)^{n+1}\big)(\Gamma_n-\gamma_n).\end{aligned}$$

The equality in (7) is attained by choosing

$$f(t)=\int_a^t\bigg(\int_a^{y_n}\Big(\cdots\int_a^{y_2}j(y_1)\,\mathrm{d}y_1\cdots\Big)\mathrm{d}y_{n-1}\bigg)\mathrm{d}y_n,$$

where

$$j(t)=\begin{cases}\gamma_n, & a\le t\le t_1=a+\dfrac{1}{\sqrt[n]{n+1}}\,(x-a),\\ \Gamma_n, & t_1\le t<x,\\ \gamma_n, & x\le t<t_2=b-\dfrac{1}{\sqrt[n]{n+1}}\,(b-x),\\ \Gamma_n, & t_2\le t\le b,\end{cases}$$

if n is odd, and

$$j(t)=\begin{cases}\gamma_n, & a\le t<t_1,\\ \Gamma_n, & t_1\le t<x,\\ \Gamma_n, & x\le t\le t_2,\\ \gamma_n, & t_2\le t\le b,\end{cases}$$

if n is even.

Remark. It is easy to find that Theorem 5 reduces to Theorem 3 or Theorem 4 if put $n = 1$ or $n = 2$, and by the way, the sharpness of inequalities (4) and (5) are proved.

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