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THE GENERALIZED HERONIAN MEAN AND ITS INEQUALITIES

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In this paper, a class of ratio inequalities for the generalized Heronian mean of two numbers are established. Some Ky Fan type inequalities are also proved. We define the generalized Heronian mean in n variables, whose properties, including Schur-convexity, are investigated.

1. INTRODUCTION AND NOTATION

Let a and b be two non-negative real numbers. The Heronian mean of a and b is defined as

$$He(a,b) = \frac{a+\sqrt{ab}+b}{3}.$$

For the "classical" Heronian mean $He(a,b)$ is known that the sharpest double-inequality of type

$$M_\alpha(a,b) \le He(a,b) \le M_\beta(a,b) \tag{1.1}$$

is given by $\alpha = \dfrac{\ln 2}{\ln 3}$ and $\beta = \dfrac{2}{3}$ (See [**1**] and [**2**], p. 350), where as usual

$$M_p(a,b) = \begin{cases} \left(\dfrac{a^p+b^p}{2}\right)^{1/p}, & p \ne 0, \\ \sqrt{ab}, & p = 0 \end{cases}$$

denotes the p-th power-mean of a and b.

In [**3**], $\widetilde{H}(a,b)$ is defined by

$$\widetilde{H}(a,b) = \frac{a+4\sqrt{ab}+b}{6}.$$

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The following double-inequality

$$M_{1/3}(a,b) \le \widetilde{H}(a,b) \le M_{1/2}(a,b) \tag{1.2}$$

is established.

Recently, Walther Janous [**4**] defined further the generalized Heronian mean

$$H_\omega(a,b) = \begin{cases} \dfrac{a+\omega\sqrt{ab}+b}{\omega+2}, & 0 \le \omega < +\infty, \\ \sqrt{ab}, & \omega = +\infty. \end{cases}$$

Clearly, $H_1(a,b) = He(a,b)$ and $H_4(a,b) = \widetilde{H}(a,b)$. The author established several inequalities for it, some of which are concerned with some well-known means such as the logarithmic mean $L(a,b)$ and the identric mean $I(a,b)$ defined as

$$L(a,b) = \begin{cases} \dfrac{a-b}{\ln a - \ln b}, & a \ne b, \\ a, & a = b \end{cases}$$

and

$$I(a,b) = \begin{cases} \dfrac{1}{e}\left(\dfrac{a^a}{b^b}\right)^{1/(a-b)}, & a \ne b, \\ a, & a = b, \end{cases}$$

respectively. The main results read as follows

Theorem A. *Let $\omega > 0$ be given. Then the optimum values α and β such that*

$$M_\alpha(a,b) \le H_\omega(a,b) \le M_\beta(a,b)$$

holds in general, are

1) *in case of $\omega \in (0,2]$: $\alpha_{\max} = \dfrac{\ln 2}{\ln(\omega+2)}$ and $\beta_{\min} = \dfrac{2}{\omega+2}$,*

2) *in case of $\omega \in [2,+\infty)$: $\alpha_{\max} = \dfrac{2}{\omega+2}$ and $\beta_{\min} = \dfrac{\ln 2}{\ln(\omega+2)}$.*

Theorem B. *The optimum numbers α and β such that*

$$H_\alpha(a,b) \le L(a,b) \le H_\beta(a,b) \tag{1.3}$$

is true in general, are $\alpha_{min} = +\infty$ and $\beta_{max} = 4$.

Theorem C. *The optimum numbers α and β such that*

$$H_\alpha(a,b) \le I(a,b) \le H_\beta(a,b) \tag{1.4}$$

is valid in general, are $\alpha_{\min} = 1$ and $\beta_{\max} = e-2$.

Remark. For the logarithmic and identric means, there are some good results. See, for example, [**4**, **7**, **9**, **11-15**] and the references cited therein.

It should be noted that the above Heronian mean and its generalizations are those of two real numbers. Now, the generalized Heronian mean in n variables will be defined.

Definition 1.1. *Let $x = (x_1, x_2, \ldots, x_n)$, $x_i \geq 0, i = 1, 2, \ldots, n$. The generalized Heronian mean in n variables $x_1, x_2, \ldots, x_n$ is defined as*

$$H_\omega(x) = H_\omega(x_1, x_2, \ldots, x_n) = \begin{cases} \dfrac{x_1 + x_2 + \cdots + x_n + \omega \sqrt[n]{x_1 x_2 \cdots x_n}}{\omega + n}, & 0 \leq \omega < +\infty, \\ \sqrt[n]{x_1 x_2 \cdots x_n}, & \omega = +\infty. \end{cases}$$

Obviously, when $n = 2$, it reduces to $H_\omega(a, b)$. Thus, the mean generalizes $H_\omega(a, b)$.

For fixed $n \geq 2$, let

$$x = (x_1, x_2, \ldots, x_n),\ y = (y_1, y_2, \ldots, y_n)$$

be two n-tuples of real numbers. Let

$$x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}, y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]},$$

be their ordered components. We give the following

Definition 1.2. ([**5**, p. 55]) *The n-tuple x is to be majorized by y (in symbols $x \prec y$), if*

$$\sum_{i=1}^{m} x_{[i]} \leq \sum_{i=1}^{m} y_{[i]},\ m = 1, 2, \ldots, n-1; \tag{1.5}$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}. \tag{1.6}$$

The Schur-convex function was introduced by I. Schur in 1923 [**5**]. It has many important applications in analytic inequalities. Hardy, Littlewood, and Pólya were also interested in some inequalities that are related to Schur-convex functions [**16**]. Its definition is following

Definition 1.3. ([**5**, p. 54]) *A real-valued function ϕ defined on a set $\Omega \subset \mathbb{R}^n$ is said to be Schur-convex function on Ω if*

$$x \prec y \text{ on } \Omega \Rightarrow \phi(x) \leq \phi(y).$$

If, in addition, $\phi(x) < \phi(y)$ whenever $x \prec y$ but x is not a permutation of y, then ϕ is said to be strictly Schur-convex on Ω. ϕ is Schur-concave function on Ω if and only if $-\phi$ is Schur-convex function; ϕ is a strictly Schur-concave function on Ω if and only if $-\phi$ is strictly Schur-convex function on Ω.

For more details the interested readers can see [**5**], [**6**], [**8**] and [**9**].

Throughout the paper we assume that $\mathbb{R}_+^n = \{x = (x_1, x_2, \ldots, x_n) \mid x_i > 0, i = 1, 2, \ldots, n\}$. The un-weighted arithmetic and geometric means of x, denoted by $A_n(x)$, $G_n(x)$, respectively, are defined as follows

$$A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i, \quad G_n(x) = \left(\prod_{i=1}^{n} x_i\right)^{1/n}.$$

Assume that $0 \le x_i < 1$, $1 \le i \le n$ and define $1-x = (1-x_1, 1-x_2, \ldots, 1-x_n)$. The symbols $A_n(1-x)$, $G_n(1-x)$ also stand for the un-weighted arithmetic, geometric means of $1 - x$, respectively.

A remarkable new counterpart of the inequality $G_n(x) \le A_n(x)$ has been published in [**17**, p. 5].

Theorem D. *If* $0 < x_i \le 1/2$, *for all* $i = 1, 2, \ldots, n$, *then*

$$\frac{G_n(x)}{G_n(1-x)} \le \frac{A_n(x)}{A_n(1-x)} \tag{1.7}$$

with equality only and only if all the x_i *are equal.*

This result, commonly referred to as the Ky Fan inequality, has stimulated an interest of many researchers. New proofs, improvements and generalizations of the inequality (1.7) have been found (For instance, see [**7**, **12**, **18**]).

The paper is organized as follows. In section 3, some ratio inequalities for the generalized Heronian mean $H_\omega(a, b)$ are established. Several "Ky Fan" type inequalities are also obtained in section 4. The properties of $H_\omega(x_1, x_2, \ldots, x_n)$, including Schur-convexity, are investigated in the final section.

2. LEMMAS

In order to verify our results, the following lemmas are necessary.

Lemma 2.1. ([**4**]) (i) *For* $\omega \in (0, 2)$ *there holds* $(\omega + 2)^4 > 4^{\omega+2}$. (ii) *For* $\omega > 2$ *the reversed inequality holds true.*

Lemma 2.2. ([**5**, p. 57; **6**, p. 259]) *Let* $f(x) = f(x_1, x_2, \ldots, x_n)$ *be symmetric and have continuous partial derivatives on* $I^n = I \times I \times \cdots \times I$ *(*n *copies), where* I *is an open interval. Then* $f : I^n \to \mathbb{R}$ *is Schur-convex if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \ge 0 \tag{2.1}$$

on I^n*. It is strictly Schur-convex if* (2.1) *is a strict inequality for* $x_i \ne x_j$, $1 \le i, j \le n$.

Since $f(x)$ is symmetric, Schur's condition, i.e. (2.1), can be reduced as [**5**, p. 57]

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \ge 0, \tag{2.2}$$

and f is strictly SCHUR-convex if (2.2) is a strict inequality for $x_1 \neq x_2$. The SCHUR's condition that guarantees a symmetric function being SCHUR-concave is the same as (2.1) or (2.2) except for the direction of the inequality.

In SCHUR's condition, the domain of $f(x)$ does not have to be a Cartesian product I^n. Lemma 2.2 remains true if we replace I^n by a set $A \subseteq R^n$ with the following properties ([**5**, p. 57]):

(i) A is convex and has a nonempty interior;

(ii) A is symmetric in the sense that $x \in A$ implies $Px \in A$ for any $n \times n$ permutation matrix P.

3. SOME RATIO INEQUALITIES OF $H_\omega(a, b)$.

In the section we establish a class of ratio inequalities of the generalized Heronian mean $H_\omega(a, b)$. All inequalities are best possible.

Theorem 3.1. *Let $\omega > 0$ be given and assume that $b_1 \geq b_2 > 0, \dfrac{a_1}{b_1} \geq \dfrac{a_2}{b_2} > 0$. Then*

(i) *in case of* $\omega \in (0, 2]$:

$$\frac{M_q(a_1, a_2)}{M_q(b_1, b_2)} \leq \frac{H_\omega(a_1, a_2)}{H_\omega(b_1, b_2)} \leq \frac{M_p(a_1, a_2)}{M_p(b_1, b_2)}\,; \tag{3.1}$$

(ii) *in case of* $\omega \in [2, \infty)$:

$$\frac{M_p(a_1, a_2)}{M_p(b_1, b_2)} \leq \frac{H_\omega(a_1, a_2)}{H_\omega(b_1, b_2)} \leq \frac{M_q(a_1, a_2)}{M_q(b_1, b_2)}\,, \tag{3.2}$$

the above equalities hold if and only if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$ $(\omega \neq 2)$, *where* $p = \dfrac{2}{\omega + 2}$, $q = \dfrac{\ln 2}{\ln(\omega + 2)}$.

Proof. Simply calculating reveals

$$\frac{H_\omega(a_1, a_2)}{H_\omega(b_1, b_2)} = \frac{a_2}{b_2} \, \frac{\dfrac{a_1}{a_2} + \omega\sqrt{\dfrac{a_1}{a_2}} + 1}{\dfrac{b_1}{b_2} + \omega\sqrt{\dfrac{b_1}{b_2}} + 1}\,, \tag{3.3}$$

and

$$\frac{M_r(a_1, a_2)}{M_r(b_1, b_2)} = \frac{a_2}{b_2} \left(\frac{\left(\dfrac{a_1}{a_2}\right)^r + 1}{\left(\dfrac{b_1}{b_2}\right)^r + 1} \right)^{1/r}. \tag{3.4}$$

In view of (3.3) and (3.4), in order to prove the theorem, we only need to compare $M_q\left(\dfrac{a_1}{a_2}, 1\right) / M_q\left(\dfrac{b_1}{b_2}, 1\right)$, $H_\omega\left(\dfrac{a_1}{a_2}, 1\right) / H_\omega\left(\dfrac{b_1}{b_2}, 1\right)$ and $M_p\left(\dfrac{a_1}{a_2}, 1\right) / M_p\left(\dfrac{b_1}{b_2}, 1\right)$. Let $\dfrac{a_1}{a_2} = x$ and $\dfrac{b_1}{b_2} = y$, then $x \geq y \geq 1$. Below, we consider two possible cases for ω.

(i) For $\omega = 2$ we are done due to $H_2(a,b) = M_{1/2}(a,b)$.

(ii) For $\omega \neq 2$. Firstly, let us compare $\dfrac{H_\omega(x,1)}{H_\omega(y,1)}$ and $\dfrac{M_{\ln 2/\ln(\omega+2)}(x,1)}{M_{\ln 2/\ln(\omega+2)}(y,1)}$ by considering the following difference

$$\begin{aligned} N(x,y) &= \big(\ln H_\omega(x,1) - \ln H_\omega(y,1)\big) - \big(\ln M_{\ln 2/\ln(\omega+2)}(x,1) - \ln M_{\ln 2/\ln(\omega+2)}(y,1)\big) \\ &= \big(\ln H_\omega(x,1) - \ln M_{\ln 2/\ln(\omega+2)}(x,1)\big) - \big(\ln H_\omega(y,1) - \ln M_{\ln 2/\ln(\omega+2)}(y,1)\big). \end{aligned}$$

Set

$$f(x) = \ln H_\omega(x,1) - \ln M_{\ln 2/\ln(\omega+2)}(x,1),\ x \geq 1.$$

For convenience, letting $x = s^{2\ln(\omega+2)}$ $(s \geq 1)$ and using the abbreviation $p = \ln(\omega+2)$, we have

$$(\ln 2) f(x) = A(s),$$

where

$$A(s) = (\ln 2)\cdot \ln(1+\omega\cdot s^p + s^{2p}) - p\cdot\ln(1+s^{2\ln 2}),\ s\geq 1.$$

Differentiating $A(s)$ with respect to s, we obtain

$$A'(s) = p\cdot(\ln 2)\cdot s^{-1}\left(\frac{\omega\cdot s^p + 2s^{2p}}{1+\omega\cdot s^p + 2s^{2p}} - \frac{2s^{2\ln 2}}{1+s^{2\ln 2}}\right),$$

i.e. (as a short simplification shows)

$$A'(s) = \frac{p\cdot(\ln 2)\cdot s^{2\ln 2-1}\cdot K(s)}{(1+\omega s^p + s^{2p})(1+s^{2\ln 2})},$$

where

$$K(s) = 2s^{2p-2\ln 2} - \omega\cdot s^p + \omega\cdot s^{p-2\ln 2} - 2.$$

But $K'(s) = s^{p-2\ln 2-1}\cdot L(s)$ with

$$L(s) = 2\,(2p - 2\ln 2)s^p - p\cdot\omega\cdot s^{2\ln 2} + \omega(p - 2\ln 2),$$

whence finally

$$L'(s) = p\cdot s^{2\ln 2-1}\cdot\big((4p - \ln 16)s^{p-2\ln 2} - 2\omega\ln 2\big).$$

We now distinguish two cases.

(1) $\omega \in (0,2)$. By Lemma 2.1, we have $L'(1) > 0$. Because of $p - 2\ln 2 < 0$ it follows $L'(s) > 0$ for $s \in (1, s_0)$ and $L'(s) < 0$ for $s \in (s_0, +\infty)$ where $s_0 = \left(\dfrac{2\omega\ln 2}{4p - \ln 16}\right)^{1/(p-2\ln 2)}$. Thus, $L(1) > 0$ and $L(s) \to -\infty$ as $s \to +\infty$ (note $p < 2\ln 2$) yield the existence of precisely one $s_1 \in (s_0, +\infty)$ such that $K(s)$ increases as $s \in (1, s_1)$ and $K(s)$ decreases as $s \in (s_1, +\infty)$. Noting $A'(1) = 0$ and $\lim_{s\to+\infty} A'(s) = 0$, we have $A'(s) \geq 0$, $s > 1$, which shows that $A(s)$ increases in $[1, +\infty)$. Namely, $f(x)$ increases as $x \in [1, \infty)$. Therefore, if $x \geq y \geq 1$, then

$f(x) \geq f(y)$. There follows that $N(x,y) \geq 0$. Simply calculation shows that the left-hand inequality of (3.1) holds.

(2) $\omega \in (2,+\infty)$. Reasoning in a similar fashion as before we have $L'(s) < 0$ as $s \in (1, s_0)$ and $L'(s) > 0$ for $s \in (s_0, +\infty)$. As in (1) there follows the existence of two interval $(1, s_2)$ and $(s_2, +\infty)$ on which $K(s)$ decreases and increases, resp., finally leading to $A(s)$ (or $f(x)$) decreases in $[1,+\infty)$. It follows that $N(x,y) \leq 0$, which implies that the right-hand inequality of (3.2) is true.

Next we consider $\dfrac{H_\omega(x,1)}{H_\omega(y,1)}$ and $\dfrac{M_{2/(\omega+2)}(x,1)}{M_{2/(\omega+2)}(y,1)}$. Let

$$\begin{aligned} E(x,y) &= \big(\ln H_\omega(x,1) - \ln H_\omega(y,1)\big) - \big(\ln M_{2/(\omega+2)}(x,1) - \ln M_{2/(\omega+2)}(y,1)\big) \\ &= \big(\ln H_\omega(x,1) - \ln M_{2/(\omega+2)}(x,1)\big) - \big(\ln H_\omega(y,1) - \ln M_{2/(\omega+2)}(y,1)\big), \end{aligned}$$

and

$$g(x) = \ln H_\omega(x,1) - \ln M_{2/(\omega+2)}(x,1), \ x \geq 1.$$

Setting $x = s^{2(\omega+2)}$ $(s \geq 1)$, we obtain

$$2g(x) = 2\ln(1+\omega\cdot s^{\omega+2}+s^{2(\omega+2)}) - (\omega+2)\ln(1+s^4) + (\omega+2)\ln 2 - 2\ln(\omega+2), s \geq 1.$$

Put

$$B(s) = 2\ln(1+\omega\cdot s^{\omega+2}+s^{2(\omega+2)}) - (\omega+2)\ln(1+s^4) + (\omega+2)\ln 2 - 2\ln(\omega+2), s \geq 1.$$

Differentiating $B(s)$ with respect to s, we have

$$B'(s) = \frac{2(\omega+2)s^3 \cdot F(s)}{(1+\omega\cdot s^{\omega+2}+s^{2(\omega+2)})\cdot(1+s^4)},$$

where $F(s) = 2s^{2\omega} - \omega \cdot s^{\omega+2} + \omega \cdot s^{\omega-2} - 2, \ s \geq 1$.

But

$$F'(s) = \omega s^{\omega-3} G(s), G(s) = 4s^{\omega+2} - (\omega+2)s^4 + \omega - 2,$$

and

$$G'(s) = 4(\omega+2)s^3(s^{\omega-2} - 1), \ s \geq 1,$$

which shows $G'(s) \leq 0$ as $\omega \in (0,2)$ and $G'(s) \geq 0$ for $\omega \in (2,+\infty)$. Therefore, due to $F'(1) = G(1) = 0$ there follows for $s \geq 1$: $F'(s) \leq 0$ for $\omega \in (0,2)$ and $F'(s) \geq 0$ as $\omega \in (2,+\infty)$. This and $F(1) = 0$ imply for $s \geq 1$: $F(s) \leq 0$ if $\omega \in (0,2)$ and $F(s) \geq 0$ if $\omega \in (2,+\infty)$. And thus, $B(s)$ (or $g(x)$) decreases in $[1,+\infty)$ if $\omega \in (0,2)$ and $B(s)$ (or $g(x)$) increases in $[1,+\infty)$ if $\omega \in (2,+\infty)$. From this, we can get the following conclusions:

(1) if $\omega \in (0,2)$, then $E(x,y) \leq 0$, which shows that the right-hand inequality of (3.1) is true.

(2) if $\omega \in (2,+\infty)$, then $E(x,y) \geq 0$ implies the left-hand inequality of (3.2).

Finally, from the process of the proof, one can easily find that the equalities hold if and only if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$.

Summarizing the above discussion, we have completed the proof of Theorem 3.1.

Theorem 3.2. *If* $b_1 \geq b_2 > 0$, $\frac{a_1}{b_1} \geq \frac{a_2}{b_2} > 0$, *then*

$$\frac{G(a_1,a_2)}{G(b_1,b_2)} \leq \frac{L(a_1,a_2)}{L(b_1,b_2)} \leq \frac{H_4(a_1,a_2)}{H_4(b_1,b_2)}\,, \tag{3.5}$$

with equality if and only if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$.

Proof. (1) In the case where $b_1 = b_2$, since the theorem reduces to Theorem B, the proof is complete.

(2) Let $b_1 > b_2$. Clearly,

$$\frac{G(a_1,a_2)}{G(b_1,b_2)} = \frac{a_2}{b_2}\,\frac{\sqrt{\dfrac{a_1}{a_2}}}{\sqrt{\dfrac{b_1}{b_2}}}\,, \qquad \frac{L(a_1,a_2)}{L(b_1,b_2)} = \frac{a_2}{b_2}\,\frac{\dfrac{(a_1/a_2)-1}{\ln(a_1/a_2)}}{\dfrac{(b_1/b_2)-1}{\ln(b_1/b_2)}}\,,$$

and

$$\frac{H_4(a_1,a_2)}{H_4(b_1,b_2)} = \frac{a_2}{b_2}\,\frac{\dfrac{a_1}{a_2}+4\sqrt{\dfrac{a_1}{a_2}}+1}{\dfrac{b_1}{b_2}+4\sqrt{\dfrac{b_1}{b_2}}+1}\,.$$

Let $\dfrac{a_1}{a_2} = x$ and $\dfrac{b_1}{b_2} = y$, then $x \geq y > 1$. Thus, the inequality (3.5) is equivalent to the following

$$\sqrt{\frac{x}{y}} \leq \frac{x-1}{\ln x}\cdot\frac{\ln y}{y-1} \leq \frac{x+4\sqrt{x}+1}{y+4\sqrt{y}+1}\,. \tag{3.6}$$

In order to prove the left-hand inequality of (3.6), we have to look at the function

$$f(x) = \frac{x-1}{\sqrt{x}\ln x},\ x > 1.$$

Letting $x = s^2$ $(s > 1)$, we have

$$f(x) = g(s) = \frac{s^2-1}{2s\ln s},\ s > 1.$$

Differentiating $g(s)$ with respect to s, we get $g'(s) = \dfrac{1}{2(s\ln s)^2}\cdot h(s)$ with $h(s) = (\ln s)\,s^2 - s^2 + \ln s + 1$. Now

$$h'(s) = 2s\ln s - s + \frac{1}{s},\ h''(s) = 1 + 2\ln s - \frac{1}{s^2};\ h'''(s) = \frac{2}{s} + \frac{2}{s^3} > 0,\ s \geq 1.$$

Therefore, due to $h'(1) = h''(1) = 0$ and $\lim_{s\to 1} g'(s) = 0$ there follows $g'(s) > 0$, which implies that $g(s)$ (or $f(x)$) increases in $(1, +\infty)$. Thus, $\frac{x-1}{\sqrt{x}\ln x} \geq \frac{y-1}{\sqrt{y}\ln y}$. And so, the left-hand inequality of (3.6) holds.

For the right-hand-inequality of (3.6), let $l(x) = \frac{(x+4\sqrt{x}+1)\ln x}{x-1}$ $(x > 1)$ and $x = s^2$ $(s > 1)$, we have

$$l(x) = e(s) = \frac{2(s^2+4s+1)\cdot \ln s}{s^2-1}\,, \ s > 1.$$

Now

$$e'(s) = \frac{2}{(s^2-1)^2}\,\phi(s),$$

where $\phi(s) = -4s^2\ln s - 4s\ln s - 4\ln s + s^3 + 4s^2 - 4 - \frac{1}{s}\,,\ s > 1.$

But

$$\begin{aligned}
\phi'(s) &= -8s\ln s - 4\ln s + 3s^2 + 4s - \frac{4}{s} + \frac{1}{s^2} - 4,\\
\phi''(s) &= -8\ln s + 6s - \frac{4}{s} + \frac{4}{s^2} - \frac{2}{s^3} - 4,\\
\phi'''(s) &= 6 - \frac{8}{s} + \frac{4}{s^2} - \frac{8}{s^3} + \frac{6}{s^4},\\
\phi^{(4)}(s) &= \frac{8}{s^2}\left(1 - \frac{1}{s} + \frac{3}{s^2} - \frac{3}{s^3}\right) > 0,\ s > 1.
\end{aligned}$$

Observing that $\phi(1) = \phi'(1) = \phi''(1) = \phi'''(1) = 0$, one can easily find that $e(s)$ (or $l(x)$) increases in $(1, +\infty)$. Thus,

$$\frac{(x+4\sqrt{x}+1)\ln x}{x-1} \geq \frac{(y+4\sqrt{y}+1)\ln y}{y-1},\ x \geq y > 1.$$

A short simplification shows that the right-hand-inequality of (3.6) holds.

From the above discussion, it is easy to verify that the equalities of (3.5) hold if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$. And so, the proof is complete.

Theorem 3.3. *If* $b_1 \geq b_2 > 0$, $\frac{a_1}{b_1} \geq \frac{a_2}{b_2} > 0$, *then*

$$\frac{H_1(a_1,a_2)}{H_1(b_1,b_2)} \leq \frac{I(a_1,a_2)}{I(b_1,b_2)} \leq \frac{H_{e-2}(a_1,a_2)}{H_{e-2}(b_1,b_2)}\,, \tag{3.7}$$

with equality if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$.

Proof. (i) In the case that $b_1 = b_2$, we are done because the theorem reduces to Theorem C.

(ii) When $b_1 > b_2$, one can easily obtain

$$\frac{H_1(a_1,a_2)}{H_1(b_1,b_2)} = \frac{a_2}{b_2}\,\frac{\frac{a_1}{a_2}+\sqrt{\frac{a_1}{a_2}}+1}{\frac{b_1}{b_2}+\sqrt{\frac{b_1}{b_2}}+1}, \qquad \frac{I(a_1,a_2)}{I(b_1,b_2)} = \frac{a_2}{b_2}\,\frac{\left(\frac{a_1}{a_2}\right)^{a_1/(a_1-a_2)}}{\left(\frac{b_1}{b_2}\right)^{b_1/(b_1-b_2)}}.$$

Let $\frac{a_1}{a_2} = x$, $\frac{b_1}{b_2} = y$, then $x \geq y > 1$. Thus, (3.7) is equivalent to the following inequality

$$\frac{H_1(x,1)}{H_1(y,1)} \leq \frac{x^{x/(x-1)}}{y^{y/(y-1)}} \leq \frac{H_q(x,1)}{H_q(y,1)}. \tag{3.8}$$

Consider the left-hand inequality of (3.8). Let

$$\rho(x) = \frac{x^{x/(x-1)}}{H_1\ (x,1)}, x > 1. \tag{3.9}$$

Taking logarithmic on (3.9) yields

$$\theta(x) = \ln\rho(x) = \frac{x}{x-1}\ln x - \ln(x+\sqrt{x}+1) - \ln 3.$$

For convenience, letting $x = s^2$ $(s > 1)$, we have

$$\theta(x) = \theta_1(s) = \frac{2s^2\ln s}{s^2-1} - \ln(s^2+s+1) - \ln 3.$$

Differentiating $\theta_1(s)$ with respect to s, we get

$$\theta_1'(s) = \frac{1}{(s^2-1)^2(s^2+s+1)} \cdot \theta_2(s),$$

where $\theta_2(s) = -4s^3\ln s - 4s^2\ln s - 4s\ln s + s^4 + 4s^3 - 4s - 1$. But

$$\begin{aligned}
\theta_2'(s) =&\ -12s^2\ln s - 8s\ln s - 4\ln s + 4s^3 + 8s^2 - 4s - 8;\\
\theta_2''(s) =&\ -24s\ln s - 8\ln s + 12s^2 + 4s - \frac{4}{s} - 12;\\
\theta_2'''(s) =&\ -24\ln s + 24s - \frac{8}{s} + \frac{4}{s^2} - 20;\\
\theta_2^{(4)}(s) =&\ 24\left(1-\frac{1}{s}\right) + \frac{8}{s^2}\left(1-\frac{1}{s}\right) > 0, s > 1.
\end{aligned}$$

Noticing $\theta_2'(1) = \theta_2''(1) = \theta_2'''(1) = 0$ and $\lim_{s\to 1}\theta_1'(1) = 0$, we have $\theta'(s) \geq 0$, $s > 1$. Therefore, $\rho(x)$ increases in $(1, +\infty)$, from which, it follows that

$$\frac{x^{x/(x-1)}}{H_1(x,1)} \geq \frac{y^{y/(y-1)}}{H_1(y,1)} \quad \text{or} \quad \frac{H_1(x,1)}{H_1(y,1)} \leq \frac{x^{x/(x-1)}}{y^{y/(y-1)}}, \quad x \geq y > 1.$$

Below, setting $w(x) = \dfrac{x^{x/(x-1)}}{H_{e-2}(x)}$ $(x > 1)$ and taking logarithm on it, we have

$$u(x) = \frac{x}{x-1}\ln x - \ln\left(x + (e-2)\sqrt{x} + 1\right) - 1,\ x > 1.$$

Putting $x = s^2$ $(s > 1)$ there follows that

$$u(x) = v(s) = \frac{2s^2 \ln s}{s^2 - 1} - \ln\left(s^2 + (e-2)s + 1\right) - 1,\ s > 1.$$

Differentiating $v(s)$ with respect to s, we obtain

$$v'(s) = \frac{1}{(s^2 + (e-2)s + 1)(s^2-1)^2} \cdot z(s),\ s > 1,$$

where $z(x) = -4s^3 \ln s - 4(e-2)s^2 \ln s - 4s\ln s + (e-2)s^4 + 4s^3 - 4s - (e-2)$. Now

$$\begin{aligned}
z'(x) = &-12s^2 \ln s - 8(e-2)s\ln s - 4\ln s + 4(e-2)s^3 + 8s^2 - 4(e-2)s - 8;\\
z''(s) = &-24s\ln s - 8(e-2)\ln s + 12(e-2)s^2 + 4s - \frac{4}{s} - 12(e-2);\\
z'''(s) = &-24\ln s + 24(e-2)s - \frac{8(e-2)}{s} + \frac{4}{s^2} - 20;\\
z^{(4)}(s) = &\left(24 + \frac{8}{s^2}\right)\left(e - 2 - \frac{1}{s}\right) < 0,\ s > 1.
\end{aligned}$$

Notice $z'''(1) = 24(e-3) < 0$, $z''(1) = z'(1) = z(1) = 0$. There follows $z(x) < 0$. And thus, $v'(s) < 0$, which means that $v(s)$ (or $u(x)$) decreases in $(1, +\infty)$. Hence $w(x)$ is decreasing for $x \in [1, +\infty)$, from this, it follows that

$$\frac{x^{x/(x-1)}}{H_{e-2}(x)} \le \frac{y^{y/(y-1)}}{H_{e-2}(y)},\ x \ge y > 1,$$

which leads to the right-hand inequality of (3.8).

From the above proof, it is clear that the equalities of (3.7) hold if and only if $\frac{a_1}{a_2} = \frac{b_1}{b_2}$.

Summarizing the above discussion, we have proven the theorem.

Remark 3.4. When $b_1 = b_2$, Theorem 3.1, Theorem 3.2 and Theorem 3.3 reduce to Theorem A, Theorem B and Theorem C, respectively. From which, all inequalities established here are best possible.

4. SOME "KY FAN" TYPE INEQUALITIES

In the section, we establish several "Ky Fan" type inequalities by use of the generalized Heronian mean $H_\omega(a, b)$.

Theorem 4.1. *If* $0 < a_1 < a_2 \le 1/2$, *then*

(i) *in case of* $\omega \in (0, 2]$:

$$\frac{M_q(a_1, a_2)}{M_q(1-a_1, 1-a_2)} \le \frac{H_\omega(a_1, a_2)}{H_\omega(1-a_1, 1-a_2)} \le \frac{M_p(a_1, a_2)}{M_p(1-a_1, 1-a_2)};$$

(ii) *in case of* $\omega \in [2, +\infty)$:

$$\frac{M_p(a_1, a_2)}{M_p(1-a_1, 1-a_2)} \le \frac{H_\omega(a_1, a_2)}{H_\omega(1-a_1, 1-a_2)} \le \frac{M_q(a_1, a_2)}{M_q(1-a_1, 1-a_2)},$$

where $p = \dfrac{2}{\omega+2}$, $q = \dfrac{\ln 2}{\ln(\omega+2)}$.

Proof. The fact that $0 < a_1 < a_2 \le 1/2$ implies that $1 - a_1 > 1 - a_2 \ge 1/2$. One can easily find that

$$\frac{H_\omega(a_1, a_2)}{H_\omega(1-a_1, 1-a_2)} = \frac{a_1}{1-a_2} \frac{\dfrac{a_2}{a_1} + \omega\sqrt{\dfrac{a_2}{a_1}} + 1}{\dfrac{1-a_1}{1-a_2} + \omega\sqrt{\dfrac{1-a_1}{1-a_2}} + 1},$$

and

$$\frac{M_r(a_1, a_2)}{M_r(1-a_1, 1-a_2)} = \frac{a_1}{1-a_2} \left(\frac{\left(\dfrac{a_2}{a_1}\right)^r + 1}{\left(\dfrac{1-a_1}{1-a_2}\right)^r + 1} \right)^{1/r}.$$

Let $\dfrac{a_2}{a_1} = x$, $\dfrac{1-a_1}{1-a_2} = y$, then $x > y > 1$ (since $f(x) = x(1-x)$ increases in $(0, 1/2]$). The rest proof of the theorem is similar to that of Theorem 3.1 and be omitted.

Theorem 4.2. *Assume that* $0 < a_1 < a_2 \le 1/2$. *Then*

$$\frac{G(a_1, a_2)}{G(1-a_1, 1-a_2)} \le \frac{L(a_1, a_2)}{L(1-a_1, 1-a_2)} \le \frac{H_4(a_1, a_2)}{H_4(1-a_1, 1-a_2)}.$$

Proof. From the condition of the theorem, it follows that $1 - a_1 > 1 - a_2 \ge 1/2$. Simply calculating shows that

$$\frac{H_\omega(a_1, a_2)}{H_\omega(1-a_1, 1-a_2)} = \frac{a_1}{1-a_2} \frac{\dfrac{a_2}{a_1} + \omega\sqrt{\dfrac{a_2}{a_1}} + 1}{\dfrac{1-a_1}{1-a_2} + \omega\sqrt{\dfrac{1-a_1}{1-a_2}} + 1},$$

and

$$\frac{L(a_1, a_2)}{L(1-a_1, 1-a_2)} = \frac{a_1}{1-a_2} \frac{\dfrac{(a_2/a_1) - 1}{\ln(a_2/a_1)}}{\dfrac{(1-a_1)/(1-a_2) - 1}{\ln\left((1-a_1)/(1-a_2)\right)}}.$$

Let $\frac{a_2}{a_1} = x$, $\frac{1-a_1}{1-a_2} = y$, we get $x \geq y > 1$ (since $f(x) = x(1-x)$ increases in $(0, 1/2]$). The rest proof of the theorem is similar to that of Theorem 3.2 and so be omitted.

REMARK 4.3. Theorem 4.2 generalizes Proposition 5 in [**7**], that is,

If $a_1, a_2 \in (0, 1/2]$, *then* $\dfrac{G(a_1, a_2)}{G(1-a_1, 1-a_2)} \leq \dfrac{L(a_1, a_2)}{L(1-a_1, 1-a_2)}$.

Using the technology similar to Theorem 4.1 (or Theorem 4.2), we can get the following

Theorem 4.4. *If* $0 < a_1 < a_2 \leq 1/2$, *then*

$$\frac{H_1(a_1, a_2)}{H_1(1-a_1, 1-a_2)} \leq \frac{I(a_1, a_2)}{I(1-a_1, 1-a_2)} \leq \frac{H_{e-2}(a_1, a_2)}{H_{e-2}(1-a_1, 1-a_2)}.$$

5. ELEMENTARY PROPERTIES OF $H_\omega(x) = H_\omega(x_1, x_2, \ldots, x_n)$

In the section, we investigate $H_\omega(x_1, x_2, \ldots, x_n)$. Some properties of it are given. In particular, the SCHUR-convexity is proved, several inequalities are established by use of the theory of majorization (See the popular book [**5**].).

Theorem 5.1. *Let* $x_i \geq 0$, $i = 1, 2, \ldots, n$. *Then*

(i) $H_\omega(x)$ *is a non-increasing function of the variable* ω, *i.e. we have*: $H_\alpha(x) \geq H_\beta(x)$ *whenever* $0 \leq \alpha \leq \beta \leq +\infty$;

(ii) $\dfrac{H_\omega(x)}{H_{\omega-1}(x)}$ $(\omega \geq 1)$ *is a non-decreasing function of the variable* ω.

Proof. (i) Differentiating $H_\omega(x)$ with respect to ω and using the arithmetic-geometric inequality, we have

$$\frac{\mathrm{d}}{\mathrm{d}\omega} H_\omega(x) = \frac{n}{(n+\omega)^2} \big(G_n(x) - A_n(x)\big) \leq 0.$$

(ii) Differentiating $\dfrac{H_\omega(x)}{H_{\omega-1}(x)}$ with respect to ω and using the arithmetic-geometric inequality again, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{H_\omega(x)}{H_{\omega-1}(x)}\right) &= \frac{1}{(n+\omega)^2} \cdot \Big(\big(nA_n(x) + \omega G_n(x)\big)\big(nA_n(x) + (\omega-1)G_n(x)\big) \\ &\quad -(n+\omega)(n+\omega-1)G_n^2(x)\Big) : \Big(nA_n(x) + (\omega-1)G_n(x)\Big)^2 \geq 0.\end{aligned}$$

Thus, the proof is complete.

By Theorem 5.1, we have $G_n(x) \leq H_\omega(x) \leq A_n(x)$, which refines the "A-G" inequality.

Theorem 5.2. *Let* $0 < x_i \leq 1/2, i = 1, 2, \ldots, n$. *Then the function* $\dfrac{H_\omega(1-x)}{H_\omega(x)}$ $(\omega \geq 0)$ *is non-decreasing function of the variable* ω.

Proof. Clearly,

$$\frac{H_\omega(1-x)}{H_\omega(x)} = \frac{nA_n(1-x)+\omega G_n(1-x)}{nA_n(x)+\omega G_n(x)}\,,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{H_\omega(1-x)}{H_\omega(x)}\right) = \frac{n\big(A_n(x)G_n(1-x)-A_n(1-x)G_n(x)\big)}{\big(nA_n(x)+\omega G_n(x)\big)^2}\,. \tag{5.1}$$

From (5.1) and Theorem D, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{H_\omega(1-x)}{H_\omega(x)}\right) \ge 0,$$

which implies that $\frac{H_\omega(1-x)}{H_\omega(x)}$ $(\omega \ge 0)$ is non-decreasing function of the variable ω.

Corollary 5.3. *Let* $0 < x_i \le 1/2,\ i = 1, 2, \ldots, n$. *Then*

$$\frac{G_n(x)}{G_n(1-x)} \le \frac{H_\omega(x)}{H_\omega(1-x)} \le \frac{A_n(x)}{A_n(1-x)}\,. \tag{5.2}$$

Remark 5.4. The inequality (5.2) refines Ky Fan inequality.

Theorem 5.5. *The function* $H_\omega(x) = H_\omega(x_1, x_2, \ldots, x_n)$ $(\omega > 0)$ *is strictly Schur-concave in* $\mathbb{R}^n_+$ *and increases with respect to* $x_i, i = 1, 2, \ldots, n$.

Proof. It is clear that $H_\omega(x)$ is symmetric and has continuous partial derivatives on $\mathbb{R}^n_+$. Differentiating $H_\omega(x)$ with respect to x_i, we have

$$\frac{\partial H_\omega(x)}{\partial x_i} = \frac{1}{\omega+n}\left(1 + \frac{\omega}{n}\,\frac{\sqrt[n]{x_1x_2\cdots x_n}}{x_i}\right), \tag{5.3}$$

which shows that $H_\omega(x)$ is increasing with respect to x_i.

To investigate the strictly Schur-convexity, By Lemma 2.2, we only need to prove

$$(x_1-x_2)\left(\frac{\partial H_\omega(x)}{\partial x_1} - \frac{\partial H_\omega(x)}{\partial x_2}\right) < 0, \quad (x_1 \ne x_2).$$

As matter of fact, when $x_1 \ne x_2$, it follows, from (5.3), that

$$\begin{aligned}(x_1-x_2)\left(\frac{\partial H_\omega(x)}{\partial x_1} - \frac{\partial H_\omega(x)}{\partial x_2}\right) &= (x_1-x_2)\frac{\omega}{n(\omega+n)}\left(\frac{\sqrt[n]{x_1x_2\cdots x_n}}{x_1} - \frac{\sqrt[n]{x_1x_2\cdots x_n}}{x_2}\right)\\ &= -\frac{\omega\cdot\sqrt[n]{x_1x_2\cdots x_n}}{n(\omega+n)x_1x_2}(x_1-x_2)^2 < 0.\end{aligned}$$

Thus, the proof is complete.

Theorem 5.6. *The function* $\frac{H_\omega(x)}{H_{\omega-1}(x)}$ $(\omega \ge 1)$ *is strictly Schur-concave in* $\mathbb{R}^n_+$.

Proof. Let $\psi(x) = \dfrac{H_\omega(x)}{H_{\omega-1}(x)}$. It is clear that $\psi(x)$ is symmetric and has continuous partial derivatives on $\mathbb{R}^n_+$. By Lemma 2.2, we only need to prove

$$(x_1 - x_2)\left(\frac{\partial\psi(x)}{\partial x_1} - \frac{\partial\psi(x)}{\partial x_2}\right) < 0, \quad (x_1 \neq x_2).$$

To this end, differentiating $\psi(x)$ with respect to x_1, we have

$$\frac{\partial\psi(x)}{\partial x_1} = \frac{G_n(x)}{(n+\omega-1)(n+\omega)H^2_{\omega-1}(x)}\left(\frac{A_n(x)}{x_1} - 1\right).$$

Similarly,

$$\frac{\partial\psi(x)}{\partial x_2} = \frac{G_n(x)}{(n+\omega-1)(n+\omega)H^2_{\omega-1}(x)}\left(\frac{A_n(x)}{x_2} - 1\right).$$

Thus, when $x_1 \neq x_2$, we get

$$\begin{aligned}(x_1 - x_2)\left(\frac{\partial\psi(x)}{\partial x_1} - \frac{\partial\psi(x)}{\partial x_2}\right) &= (x_1 - x_2)\frac{A_n(x)G_n(x)}{(n+\omega-1)(n+\omega)H^2_{\omega-1}(x)}\left(\frac{1}{x_1} - \frac{1}{x_2}\right) \\ &= -\frac{A_n(x)G_n(x)}{(n+\omega-1)(n+\omega)H^2_{\omega-1}(x)}\,\frac{(x_1-x_2)^2}{x_1x_2} < 0.\end{aligned}$$

The proof of the theorem is complete.

Corollary 5.7. *Let* $x_i > 0$, $i = 1, 2, \dots, n$, $\sum_{i=1}^n x_i = 1$ *and* $\omega > 1$. *Then*

$$\frac{H_\omega(x)}{H_\omega(1-x)} \leq \frac{H_{\omega-1}(x)}{H_{\omega-1}(1-x)}. \tag{5.4}$$

Proof. By [**10**], it follows that $\left(\dfrac{1-x_1}{n-1}, \dfrac{1-x_2}{n-1}, \dots, \dfrac{1-x_n}{n-1}\right) \prec (x_1, x_2, \dots, x_n)$. Using Theorem 5.6, we get (5.4).

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