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ON A SUM INVOLVING POWERS OF THE PRIME COUNTING FUNCTION

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An asymptotic formula is derived for the sum of powers of reciprocals of $\pi(n)$, where $\pi(x)$ denotes the number of primes not exceeding x. This is an extension of previous results of L. PANAITOPOL and A. IVIĆ.

1. INTRODUCTION

Denote by $\pi(x)$ the number of primes not exceeding x. J-M. DE KONINCK and A. IVIĆ [**1**, Theorem 9.1] proved that

$$\sum_{2\le n\le x} \frac{1}{\pi(n)} = \frac{1}{2}\log^2 x + O(\log x). \tag{1}$$

This asymptotic formula is obtained as a consequence of the prime number theorem

$$\pi(x) \sim \frac{x}{\log x} \quad (x \to +\infty). \tag{2}$$

In 2000, L. PANAITOPOL [**3**] improved (1) to

$$\sum_{2\le n\le x} \frac{1}{\pi(n)} = \frac{1}{2}\log^2 x - \log x - \log\log x + O(1). \tag{3}$$

This asymptotic formula is a consequence of the following results which is due by L. PANAITOPOL [**3**]

$$\frac{1}{\pi(x)} = \frac{1}{x}\left(\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \cdots - \frac{k_m\big(1+\alpha_m(x)\big)}{\log^m x}\right), \tag{4}$$

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where $\alpha_m(x) <<_m \dfrac{1}{\log x}$, and the constants $k_1, k_2, \ldots, k_m$ are defined by the recurrence relation

$$k_n + 1!\,k_{n-1} + 2!\,k_{n-2} + \cdots + (n-1)!\,k_1 = n \cdot n! \qquad (n \in \mathbb{N}). \tag{5}$$

Easy calculations give

$$k_1 = 1,\ k_2 = 3,\ k_3 = 13,\ k_4 = 71,\ k_5 = 461,\ k_6 = 3447,\ \text{etc.}$$

In [**3**] L. Panaitopol gives the following formula for k_n

$$k_n = \det(a_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le n}} \tag{6}$$

with

$$a_{i,j} = \begin{cases} (n+1-i) \cdot (n+1-i)! & \text{if } j = 1 \text{ and } 1 \le i \le n, \\ 0 & \text{if } 2 \le j < i \le n, \\ (j-i)! & \text{if } 1 \le i \le j \le n \text{ and } j > 1. \end{cases} \tag{7}$$

To prove (3), L. Panaitopol uses (4) for $m = 2$.

Recently, using (4), A. Ivić gives the following further improvement of (2), for any fixed integer $m \ge 2$

$$\sum_{2 \le n \le x} \frac{1}{\pi(n)} = \frac{1}{2} \log^2 x - \log x - \log\log x + C + \frac{k_2}{\log x} + \frac{k_3}{2\log^2 x} + \cdots + \frac{k_m}{(m-1)\log^{m-1} x} + O\left(\frac{1}{\log^m x}\right), \tag{8}$$

where C is an absolute constant, and $k_2, \ldots, k_m$ are the constants defined by (5).

In this paper, we give an asymptotic formula for the more general sum $\displaystyle\sum_{2 \le n \le x} \left(\frac{1}{\pi(n)}\right)^r$ for any integer $r \ge 2$.

This result contained in the Theorem stated in the following section is obtained by using (4).

2. STATEMENT OF THE THEOREM

Let $r \ge 2$ be an integer. Then the serie $\displaystyle\sum_{n \ge 2} \left(\frac{1}{\pi(n)}\right)^r$ converges. Indeed, from the prime number theorem (2), we have

$$\left(\frac{1}{\pi(n)}\right)^r \sim \frac{\log^r n}{n^r} \qquad (n \to +\infty).$$

Let us denote by C_r its sum

(9) $$C_r = \sum_{n\geq 2} \left(\frac{1}{\pi(n)}\right)^r.$$

Let $(\alpha_{r,n})_{n\geq 0}$ be the sequence of real numbers defined by the following equality in $\mathbb{R}[[X]]$

(10) $$\Big(1 - X - \sum_{n\geq 1} k_n X^{n+1}\Big)^r = \sum_{n\geq 0} \alpha_{r,n} X^n.$$

Easy calculations give

$$\alpha_{r,0} = 1, \quad \alpha_{r,1} = -r, \quad \alpha_{r,2} = \frac{1}{2} r\,(r-3), \quad \alpha_{r,3} = -\frac{1}{6}\, r\left(r^2 - 9r + 26\right),$$
$$\alpha_{r,4} = \frac{1}{24}\, r\left(r^3 - 18r^2 + 131r - 426\right), \quad \text{etc.}$$

By setting

$$b_n = \begin{cases} 1 & \text{if } n = 0, \\ -1 & \text{if } n = 1, \\ -k_{n-1} & \text{if } n \geq 2, \end{cases}$$

one has

$$\alpha_{r,n} = \sum_{\substack{(i_1, i_2, \ldots, i_r) \in \mathbb{Z}_+^r \\ i_1 + i_2 + \cdots + i_r = n}} b_{i_1} b_{i_2} \cdots b_{i_r}.$$

Theorem. *For any fixed integer $m \geq 2$, we have*

(11) $$\sum_{2\leq n\leq x} \left(\frac{1}{\pi\,(n)}\right)^r = C_r + \frac{1}{x^{r-1}} \Bigg(\lambda_{r,r} \log^r x + \cdots + \lambda_{r,1} \log x + \lambda_{r,0} + \frac{\mu_{r,1}}{\log x} + \frac{\mu_{r,2}}{\log^2 x} + \cdots + \frac{\mu_{r,m-1}}{\log^{m-1} x} + O\left(\frac{1}{\log^m x}\right)\Bigg),$$

where

(12) $$\lambda_{r,k} = -\sum_{s=k}^{r} \frac{s!}{k!\,(r-1)^{s-k+1}}\, \alpha_{r,r-s} \quad \textit{for } 0 \leq k \leq r,$$

(13) $$\mu_{r,k} = \sum_{s=1}^{k} \frac{(k-1)!}{(s-1)!\,(1-r)^{k-s+1}}\, \alpha_{r,r+s}, \quad \textit{for } 1 \leq k \leq m-1.$$

When $m = 6$, the Theorem gives for $r = 2$

$$\sum_{2\leq n\leq x} \left(\frac{1}{\pi\,(n)}\right)^2 = C_2 + \frac{1}{x}\Bigg(-\log^2 x + 1 + \frac{4}{\log x} + \frac{15}{\log^2 x} + \frac{80}{\log^3 x} + \frac{505}{\log^4 x} + \frac{3732}{\log^5 x} + O\left(\frac{1}{\log^6 x}\right)\Bigg)$$

and for $r=3$

$$\sum_{2\leq n\leq x}\left(\frac{1}{\pi(n)}\right)^3 = C_3+\frac{1}{x^2}\left(-\frac{1}{2}\log^3 x+\frac{3}{4}\log^2 x+\frac{3}{4}\log x+\frac{19}{8}+\frac{21}{2\log x}\right.$$
$$\left.+\frac{237}{4\log^2 x}+\frac{1583}{4\log^3 x}+\frac{24219}{8\log^4 x}+\frac{104091}{4\log^5 x}+O\left(\frac{1}{\log^6 x}\right)\right).$$

The proof of Theorem uses essentially the two following lemmas.

3. LEMMAS

Let $m\geq 2$ an integer. Since $r\geq 2$, the integral

$$L_q := \int_x^{+\infty}\frac{\log^q t}{t^r}\,\mathrm{d}t \quad (x>0) \tag{14}$$

converges for any $q\in\mathbb{Z}$.

When $q\leq r$ and $x\geq 3$, the function $x\mapsto \dfrac{\log^q x}{x^r}$ is decreasing and positive. Consequently, we can write

$$\sum_{n>x}\frac{\log^q n}{n^r} = L_q+O\left(\frac{\log^q x}{x^r}\right) = L_q+O\left(\frac{1}{x^{r-1}\log^m x}\right). \tag{15}$$

Also, we have

Lemma 1. *For any integer $s\geq 0$, we have*

$$L_s=\int_x^{+\infty}\frac{\log^s t}{t^r}\,\mathrm{d}t=\frac{1}{x^{r-1}}\sum_{k=0}^{s}\beta_{r,s,k}\log^k x, \tag{16}$$

with $\beta_{r,s,k}=\dfrac{s!}{k!\,(r-1)^{s-k+1}}$, *for* $0\leq k\leq s$.

Lemma 2. *For any integer $s\in\{1,2,\ldots,m-1\}$, we have*

$$L_{-s}=\int_x^{+\infty}\frac{1}{t^r\log^s t}\,\mathrm{d}t=\frac{1}{x^{r-1}}\sum_{k=s}^{m-1}\frac{\gamma_{r,s,k}}{\log^k x}+O\left(\frac{1}{x^{r-1}\log^m x}\right), \tag{17}$$

with $\gamma_{r,s,k}=-\dfrac{(k-1)!}{(s-1)!\,(1-r)^{k-s+1}}$, *for* $k\geq s$.

It follows from $(14),(15),(16)$ and (17) that for $s\in\{0,1,2,\ldots,r\}$, we have

$$\sum_{n>x}\frac{\log^s n}{n^r}=\frac{1}{x^{r-1}}\sum_{k=0}^{s}\beta_{r,s,k}\log^k x+O\left(\frac{1}{x^{r-1}\log^m x}\right) \tag{18}$$

and for $s \in \{1, 2, \ldots, m-1\}$, we have

$$\sum_{n>x} \frac{1}{n^r \log^s n} = \frac{1}{x^{r-1}} \sum_{k=s}^{m-1} \frac{\gamma_{r,s,k}}{\log^k x} + O\left(\frac{1}{x^{r-1} \log^m x}\right). \tag{19}$$

Proofs of Lemmas. For $q \in \mathbb{Z}$, an integration by parts gives

$$L_q = \frac{\log^q x}{(r-1)\, x^{r-1}} + \frac{q}{r-1} L_{q-1}. \tag{20}$$

To prove Lemma 1, we use (20) when $q = k$, where $k \geq 1$, with

$$I_q := \frac{(r-1)^q}{q!} L_q \quad \text{for } q \geq 0. \tag{21}$$

Then (20) gives

$$I_k - I_{k-1} = \frac{(r-1)^{k-1}}{k!} \frac{\log^k x}{x^{r-1}}. \tag{22}$$

For $s \geq 0$, one deduces from (14), (21) and (22)

$$\begin{aligned} L_s &= \frac{s!}{(r-1)^s} I_s = \frac{s!}{(r-1)^s} \left(I_0 + \sum_{k=1}^{s} (I_k - I_{k-1}) \right) \\ &= \frac{s!}{(r-1)^s} \left(\frac{1}{(r-1)\, x^{r-1}} + \sum_{k=1}^{s} \frac{(r-1)^{k-1}}{k!} \frac{\log^k x}{x^{r-1}} \right) \\ &= \frac{1}{x^{r-1}} \sum_{k=0}^{s} \beta_{r,s,k} \log^k x. \end{aligned}$$

with $\beta_{r,s,k} = \dfrac{s!}{k!\,(r-1)^{s-k+1}}$, for $0 \leq k \leq s$, and the Lemma 1 follows.

To prove Lemma 2, we use (20) with $q = -k$, where $k \geq 1$, with

$$J_q := -\frac{(q-1)!}{(1-r)^q} L_{-q} \quad \text{for } q \geq 1. \tag{23}$$

Then (20) gives

$$J_k - J_{k+1} = \frac{(k-1)!}{(1-r)^{k+1}} \frac{1}{x^{r-1} \log^k x}. \tag{24}$$

For $s \in \{1, 2, \ldots, m-1\}$, one deduces from (14), (23) and (24)

$$\begin{aligned} L_{-s} &= -\frac{(1-r)^s}{(s-1)!} J_s = -\frac{(1-r)^s}{(s-1)!} \left(\sum_{k=s}^{m-1} (J_k - J_{k+1}) + J_m \right) \\ &= \frac{1}{x^{r-1}} \sum_{k=s}^{m-1} \frac{\gamma_{r,s,k}}{\log^k x} + O\left(J_m\right), \end{aligned}$$

with $\gamma_{r,s,k} = -\dfrac{(k-1)!}{(s-1)!\,(1-r)^{k-s+1}}$, for $k \geq s$.

Since

$$J_m = O\left(\int_x^{+\infty} \frac{1}{t^r \log^m t}\,\mathrm{d}t\right) = O\left(\frac{1}{\log^m x}\int_x^{+\infty} \frac{1}{t^r}\,\mathrm{d}t\right) = O\left(\frac{1}{x^{r-1}\log^m x}\right).$$

The Lemma 2 is then proved.

4. PROOF OF THE THEOREM

From (4) and (10), we get for any fixed $m \geq 2$

$$\begin{aligned}
\left(\frac{1}{\pi(n)}\right)^r &= \frac{\log^r n}{n^r}\left(1 - \frac{1}{\log n} - \frac{k_1}{\log^2 n} - \cdots - \frac{k_{m+r-2}}{\log^{m+r-1} n} + O\left(\frac{1}{\log^{m+r} n}\right)\right)^r \\
&= \frac{\log^r n}{n^r}\left(\alpha_{r,0} + \frac{\alpha_{r,1}}{\log n} + \frac{\alpha_{r,2}}{\log 2n} + \cdots + \frac{\alpha_{r,m+r-1}}{\log^{m+r-1} n} + O\left(\frac{1}{\log^{m+r} n}\right)\right) \\
&= \sum_{s=0}^{r} \alpha_{r,r-s}\frac{\log^s n}{n^r} + \sum_{s=1}^{m-1} \frac{\alpha_{r,r+s}}{n^r \log^s n} + O\left(\frac{1}{n^r \log^m n}\right).
\end{aligned}$$

By summation one obtains for $x \geq 3$

$$\begin{aligned}
\sum_{2\leq n\leq x}\left(\frac{1}{\pi(n)}\right)^r = C_r - \sum_{n>x}\left(\frac{1}{\pi(n)}\right)^r &= C_r - \sum_{s=0}^{r}\alpha_{r,r-s}\left(\sum_{n>x}\frac{\log^s n}{n^r}\right) \\
&\quad - \sum_{s=1}^{m-1}\alpha_{r,r+s}\sum_{n>x}\frac{1}{n^r\log^s n} + O\left(\sum_{n>x}\frac{1}{n^r\log^m n}\right).
\end{aligned}$$

Using (18) and (19), we get

$$\begin{aligned}
\sum_{2\leq n\leq x}\left(\frac{1}{\pi(n)}\right)^r &= C_r - \frac{1}{x^{r-1}}\sum_{s=0}^{r}\sum_{k=0}^{s}\alpha_{r,r-s}\beta_{r,s,k}\log^k x \\
&\quad + \frac{1}{x^{r-1}}\sum_{s=1}^{m-1}\sum_{k=s}^{m-1}\frac{\alpha_{r,r+s}\gamma_{r,s,k}}{\log^k x} + O\left(\frac{1}{x^{r-1}\log^m x}\right).
\end{aligned}$$

Finally, we obtain

$$\sum_{2\leq n\leq x}\left(\frac{1}{\pi(n)}\right)^r = C_r + \frac{1}{x^{r-1}}\left(\sum_{k=0}^{r}\lambda_{r,k}\log^k x + \sum_{k=1}^{m-1}\frac{\mu_{r,k}}{\log^k x} + O\left(\frac{1}{\log^m x}\right)\right),$$

where $\lambda_{r,k}$ and $\mu_{r,k}$ are defined by (12) and (13), and this completes the proof.

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