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SOME INEQUALITIES FOR ALTERNATING KUREPA'S FUNCTION

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In this paper we consider alternating Kurepa's function $A(z)$. We give some recurrent relations for alternating Kurepa's function via appropriate sequences of rational functions and gamma function. Also we give some inequalities for the real part of alternating Kurepa's function $A(x)$ for values of argument $x > -2$. The obtained results are analogous to results from the author's paper [**5**].

1. ALTERNATING KUREPA'S FUNCTION $A(z)$

R. Guy considered, in the book [**3**] (p. 100.), the function of alternating left factorial as an alternating sum of factorials

(1) $$A(n) = \sum_{i=1}^{n} (-1)^{n-i}\, i!\,.$$

Sum (1) corresponds to the sequence $A005165$ in [**6**]. An analytical extension of the function (1) over the set of complex numbers is determined by the integral

(2) $$A(z) = \int_{0}^{\infty} e^{-t}\, \frac{t^{z+1} - (-1)^z t}{t+1}\, \mathrm{d}t,$$

which converges for $\operatorname{Re} z > 0$ [**4**]. For function $A(z)$ we use the term *alternating Kurepa's function.* It is easily verified that alternating Kurepa's function is a solution of the functional equation:

(3) $$A(z) + A(z-1) = \Gamma(z+1).$$

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Let us observe that since $A(z-1)=\Gamma(z+1)-A(z)$, it is possible to make the analytical continuation of alternating KUREPA's function $A(z)$ for $\operatorname{Re} z \le 0$. In that way, the alternating KUREPA's function $A(z)$ is a meromorphic function with simple poles at $z=-n$ $(n\ge 2)$ [**4**].

Let us emphasize that in the following consideration, in the sections **2.** and **3.**, it is sufficient to use only fact that function $A(z)$ is a solution of the functional equation (3). In section **4.** we give some inequalities for the real part of alternating KUREPA's function $A(x)$ for values of argument $x>-2$.

2. REPRESENTATION OF THE ALTERNATING KUREPA'S FUNCTION VIA SEQUENCES OF POLYNOMIALS AND GAMMA FUNCTION

Let us introduce a sequences of polynomials:

$$\mathfrak{p}_n(z)=(z-n+1)\mathfrak{p}_{n-1}(z)+(-1)^n, \tag{4}$$

with $\mathfrak{p}_0(z)=1$. Analogously to results from [**2**], the following statements are true:

Lemma 2.1. *For each $n\in\mathbb{N}$ and $z\in\mathbb{C}$ we have explicitly* :

$$\mathfrak{p}_n(z)=(-1)^n\left(1+\sum_{j=0}^{n-1}\prod_{i=0}^{j}(-1)^{j-1}(z-n+i+1)\right). \tag{5}$$

Theorem 2.2. *For each $n\in\mathbb{N}$ and $z\in\mathbb{C}\backslash\big(\mathbb{Z}^-\cup\{0,1,2,\dots,n-2\}\big)$ is valid*

$$A(z)=(-1)^n A(z-n)+\mathfrak{p}_{n-1}(z)\cdot\Gamma(z-n+2). \tag{6}$$

3. REPRESENTATION OF THE ALTERNATING KUREPA'S FUNCTION VIA SEQUENCES OF RATIONAL FUNCTIONS AND GAMMA FUNCTION

Let us observe that on the basis of a functional equation for the gamma function $\Gamma(z+1)=z\Gamma(z)$, it follows that the alternating KUREPA's function is solution of the following functional equation:

$$A(z+1)-zA(z)-(z+1)A(z-1)=0. \tag{7}$$

For $z\in\mathbb{C}\backslash\{-1\}$, based on (7), we have

$$A(z-1)=-\frac{z}{z+1}A(z)+\frac{1}{z+1}A(z+1)=\big(-1-\mathfrak{r}_1(z)\big)A(z)-\mathfrak{r}_1(z)A(z+1), \tag{8}$$

Letters $\mathfrak{p}$, $\mathfrak{r}$, $\mathfrak{g}$ are printed in the funny italic TEX font.

for rational function $\Upsilon_1(z) = -\dfrac{1}{z+1}$ over $\mathbb{C}\backslash\{-1\}$. Next, for $z \in \mathbb{C}\backslash\{-1,0\}$, based on (7), we obtain

$$\begin{aligned} A(z-2) &= \frac{1}{z}A(z) - \frac{z-1}{z}A(z-1) \\ &\underset{(8)}{=} \frac{1}{z}A(z) - \frac{z-1}{z}\Big(-\frac{z}{z+1}A(z) + \frac{1}{z+1}A(z+1)\Big) \\ &= \frac{z^2+1}{z(z+1)}A(z) - \frac{z-1}{z(z+1)}A(z+1) = \big(1-\Upsilon_2(z)\big)A(z) - \Upsilon_2(z)A(z+1), \end{aligned} \tag{9}$$

for rational function $\Upsilon_2(z) = \dfrac{z-1}{z(z+1)}$ over $\mathbb{C}\backslash\{-1,0\}$. Thus, for values $z \in \mathbb{C}\backslash\{-1,0,1,\ldots,n-2\}$, based on (7), by mathematical induction it is true

$$A(z-n) = \big((-1)^n - \Upsilon_n(z)\big)A(z) - \Upsilon_n(z)A(z+1), \tag{10}$$

for rational function $\Upsilon_n(z)$ over $\mathbb{C}\backslash\{-1,0,1,\ldots,n-2\}$ which fulfill the recurrent relation

$$\Upsilon_n(z) = -\frac{z-n+1}{z-n+2}\Upsilon_{n-1}(z) + \frac{1}{z-n+2}\Upsilon_{n-2}(z), \tag{11}$$

with different initial functions $\Upsilon_{1,2}(z)$.

Based on the previous consideration we can conclude:

Lemma 3.1. *For each $n \in \mathbb{N}$ and $z \in \mathbb{C}\backslash\{-1,0,1,\ldots,n-2\}$ let the rational function $\Upsilon_n(z)$ be determined by the recurrent relation* (11) *with initial functions $\Upsilon_1(z) = -\dfrac{1}{z+1}$ and $\Upsilon_2(z) = \dfrac{z-1}{z(z+1)}$. Thus the sequences $\Upsilon_n(z)$ has an explicit form*

$$\Upsilon_n(z) = (-1)^{n-1}\Bigg(\sum_{j=1}^{n}\prod_{i=1}^{j}\frac{(-1)^j}{z+2-i}\Bigg). \tag{12}$$

Theorem 3.2. *For each $n \in \mathbb{N}$ and $z \in \mathbb{C}\backslash\{-1,0,1,\ldots,n-2\}$ we have*

$$A(z) = (-1)^n\Big(A(z-n) + \Upsilon_n(z)\cdot\Gamma(z+2)\Big). \tag{13}$$

4. SOME INEQUALITIES FOR THE REAL PART OF ALTERNATING KUREPA'S FUNCTION

In this section we consider alternating KUREPA's function $A(x)$, given by an integral representation (2), for values of argument $x > -2$. The real and imaginary parts of the function $A(x)$ are represented by

$$\operatorname{Re} A(x) = \int_0^\infty e^{-t}\,\frac{t^{x+1} - \cos(\pi x)\,t}{t+1}\,\mathrm{d}t \tag{14}$$

and

$$\operatorname{Im} A(x) = -\int_0^\infty e^{-t}\,\frac{\sin(\pi x)\,t}{t+1}\,\mathrm{d}t. \tag{15}$$

In this section we give some inequalities for the real part of alternating KUREPA's function $A(x)$ for values of argument $x > -2$. The following statements are true:

Lemma 4.1. *The function*

$$\beta(x) = \int_0^\infty e^{-t}\,\frac{t^{x+1}}{t+1}\,\mathrm{d}t, \tag{16}$$

over set $(-2, \infty)$ *is positive, convex and fulfill an inequality*

$$\beta(x) \geq \beta(x_0) = 0.401\,855\ \ldots\ , \tag{17}$$

with equality in the point $x_0 = -0.108\,057\ \ldots$.

Proof. For positive function $\beta(x) \in C^2(-2, \infty)$, on the basis of (16), the condition of convexity $\beta''(x) > 0$ is true. Next, based on (16), we can conclude $\lim\limits_{\varepsilon \to 0+} \beta(-2+\varepsilon) = +\infty$ and $\lim\limits_{x \to +\infty} \beta(x) = +\infty$. Therefore, we can conclude that exists exactly one minimum $x_0 \in (-2, +\infty)$. Using standard numerical methods it is easily determined $x_0 = -0.108\,057\ \ldots$ and $\beta(x_0) = 0.401\,855\ \ldots$. □

Lemma 4.2. *The function*

$$\gamma(x) = \int_0^\infty e^{-t}\,\frac{\cos(\pi x)\,t}{t+1}\,\mathrm{d}t, \tag{18}$$

over set $(-2, \infty)$, *is determined with*

$$\gamma(x) = \big(1 + e\,\mathrm{Ei}(-1)\big) \cdot \cos(\pi x) = 0.403\,652\ \ldots\ \cdot \cos(\pi x). \tag{19}$$

where $\mathrm{Ei}(t) = \displaystyle\int_{-\infty}^{t} \frac{e^u}{u}\,\mathrm{d}u$ $(t < 0)$ *is function of exponential integral* ([**1**], 8.211-1).

Lemma 4.3. *The function* $\operatorname{Re} A(x)$, *over set* $(-2, \infty)$, *is determined as difference*

$$\operatorname{Re} A(x) = \beta(x) - \gamma(x) \tag{20}$$

and has two roots $x_1 = -0.015\,401\ \ldots$ *and* $x_2 = 0$. *The function* $\operatorname{Re} A(x)$ *is positive over set*

$$D_1 = (-2, x_1) \cup (0, \infty) \tag{21}$$

and negative over set

$$D_2 = (x_1, 0). \tag{22}$$

Proof. Let $\beta(x)$ be function from lemma 4.1 and let $\gamma(x)$ be function from lemma 4.2. For value $x_2 = 0$ it is true $\beta(x_2) = \gamma(x_2) = 0.403\,652\ldots$, ie. value $x_2 = 0$ is a root of function $\mathrm{Re}\,A(x)$. Let us prove that function $\mathrm{Re}\,A(x)$ has exactly one root $x_1 \in (x_0, x_2)$, where $x_0 = -0.108\,057\ldots$ is value from lema 4.1. It is true $\beta(x_0) = 0.401\,855\ldots > 0.380\,061\ldots = \gamma(x_0)$. Let us notice that $\beta(x)$ is convex and increasing function over set (x_0, x_2) and let us notice that $\gamma(x)$ is concave and increasing function over same set (x_0, x_2). Therefore, we can conclude that function $\mathrm{Re}\,A(x)$ has exactly one root $x_1 \in (x_0, x_2)$. Using numerical methods we can determined $x_1 = -0.015\,401\ldots$. On the basis of the graphs of the functions $\beta(x)$ and $\gamma(x)$ we can conclude that function $\mathrm{Re}\,A(x)$ has exactly two roots x_1 and x_2 over set $(-2, \infty)$. Hence, the sets D_1 and D_2 are correctly determined. □

Lema 4.4. *For* $x \in (-1, 1 + x_1] \cup [1, \infty)$ *it is true*

$$\Gamma(x+1) \geq \mathrm{Re}\,A(x), \tag{23}$$

while the equality is true for $x = 1 + x_1$ *or* $x = 1$.

Proof. For $x > -1$ it is true

$$\Gamma(x+1) \geq \mathrm{Re}\,A(x) = \Gamma(x+1) - \mathrm{Re}\,A(x-1) \iff \mathrm{Re}\,A(x-1) \geq 0. \tag{24}$$

Right side of the previous equivalence is true for $x - 1 \in (-2, x_1] \cup [0, \infty)$, ie. $x \in (-1, 1 + x_1] \cup [1, \infty)$. □

In the following considerations let us denote $\mathbf{E}_a = (a, a+2+x_1] \cup [a+2, \infty)$ for fixed $a \geq -1$.

Corollary 4.5. *For fixed* $k \in \mathbb{N}$ *and values* $x \in \mathbf{E}_k$ *following inequality is true*:

$$\frac{\mathrm{Re}\,A(x-k-1)}{\Gamma(x-k)} \leq 1, \tag{25}$$

while the equality is true for $x = k+2+x_1$ *or* $x = k+2$.

In the next two proofs of theorems which follows we use the auxiliary sequences of functions

$$\mathfrak{g}_k(x) = \sum_{i=0}^{k-1} (-1)^{k+i}\Gamma(x+1-i) \qquad (k \in \mathbb{N}), \tag{26}$$

for values $x > k-2$. Let us notice that for $x > k-2$ it is true

$$\mathfrak{g}_k(x) = \Gamma(x+2) \cdot \mathfrak{r}_k(x). \tag{27}$$

Therefore $(-1)^k \cdot \mathfrak{r}_k(x)$ are positive functions for $x \geq k+1$. Then, the following statements are true:

Theorem 4.6 *For fixed odd number* $k = 2n+1 \in \mathbb{N}$ *and values* $x \geq k+1$ *the following double inequality is true*:

$$\frac{\mathfrak{p}_k(x)}{\mathfrak{p}_k(x)+1} \cdot \big(-\mathfrak{r}_k(x)\big) \leq \frac{\mathrm{Re}\,A(x)}{\Gamma(x+2)} < \big(-\mathfrak{r}_k(x)\big), \tag{28}$$

while the equality is true for $x = k+1$.

Proof. Based on lemma 4.3, using theorem 3.2, the following inequality is true:

$$\operatorname{Re} A(x) \leq -\, \mathfrak{g}_{2n+1}(x), \tag{29}$$

for values $x \in \mathbf{E}_{k-2}$. On the other hand, based on (25), for values $x \in \mathbf{E}_{k-1}$ we can conclude

$$\begin{aligned}\frac{\operatorname{Re} A(x)}{\mathfrak{g}_{2n+1}(x)} &= -1 + \frac{\operatorname{Re} A(x-2n-1)}{\mathfrak{g}_{2n+1}(x)} = -1 + \frac{\operatorname{Re} A(x-2n-1)}{\Gamma(x-2n)(\mathfrak{p}_{2n+1}(x)+1)} \\ &= -1 + \frac{\operatorname{Re} A(x-2n-1)/\Gamma(x-2n)}{\mathfrak{p}_{2n+1}(x)+1} \leq -\frac{\mathfrak{p}_{2n+1}(x)}{\mathfrak{p}_{2n+1}(x)+1}\,.\end{aligned} \tag{30}$$

From (29) and (30), using (27), the double inequality (28) follows for values $x \geq k+1$. □

Theorem 4.7. *For fixed even number $k = 2n \in \mathbb{N}$ and values $x \geq k+1$ the following double inequality is true*:

$$\mathfrak{r}_k(x) < \frac{\operatorname{Re} A(x)}{\Gamma(x+2)} \leq \frac{\mathfrak{p}_k(x)}{\mathfrak{p}_k(x)-1} \cdot \mathfrak{r}_k(x), \tag{31}$$

while the equality is true for $x = k+1$.

Proof. Based on lemma 4.3, using theorem 3.2, the following inequality is true:

$$\operatorname{Re} A(x) \geq \mathfrak{g}_{2n}(x), \tag{32}$$

for values $x \in \mathbf{E}_{k-2}$. On the other hand, based on (25), for values $x \in \mathbf{E}_{k-1}$ we can conclude

$$\begin{aligned}\frac{\operatorname{Re} A(x)}{\mathfrak{g}_{2n}(x)} &= 1 + \frac{\operatorname{Re} A(x-2n)}{\mathfrak{g}_{2n}(x)} = 1 + \frac{\operatorname{Re} A(x-2n)}{\Gamma(x-2n+1)(\mathfrak{p}_{2n}(x)-1)} \\ &= 1 + \frac{\operatorname{Re} A(x-2n)/\Gamma(x-2n+1)}{\mathfrak{p}_{2n}(x)-1} \leq \frac{\mathfrak{p}_{2n}(x)}{\mathfrak{p}_{2n}(x)-1}.\end{aligned} \tag{33}$$

From (32) and (33), using (27), the double inequality (31) follows for values $x \geq k+1$. □

Corollary 4.8. *For fixed number $k \in \mathbb{N}$ and values $x \geq k+1$ the following double inequality is true*:

$$\mathfrak{r}_k(x) < (-1)^k \frac{\operatorname{Re} A(x)}{\Gamma(x+2)} \leq \frac{\mathfrak{p}_k(x)}{\mathfrak{p}_k(x)-(-1)^k} \cdot \mathfrak{r}_k(x), \tag{34}$$

while the equality is true for $x = k+1$.

Corollary 4.8. *On the basis of theorems* 4.6 *and* 4.7 *we can conclude*

$$\lim_{x\to\infty} \frac{\operatorname{Re} A(x)}{\Gamma(x+2)} = 0 \quad \textit{and} \quad \lim_{x\to\infty} \frac{\operatorname{Re} A(x)}{\Gamma(x+1)} = 1. \tag{35}$$

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