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GENERALIZED MOMENTS FOR THE SEQUENTIAL LAW

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We investigate asymptotic behavior of the generalized moments $E\big(X_n^\mu \ell(X_n)\big)$ for the sequential law under assumption that $\lim\limits_n \dfrac{1}{n} E(X_n)$ exists, where ℓ is a slowly varying function and $\mu \in \mathbb{R}^+$.

1. INTRODUCTION

Let a set of random variables (X_n) be defined by

$$P\{X_n = k\} = p_{nk} > 0,\ 1 \le k \le n; \quad \sum_{k\le n} p_{nk} = 1. \tag{1}$$

Accordingly, the expectation is $E(X_n) := \sum\limits_{k\le n} kp_{nk}$ and moments of the m-th order are $E(X_n^m) := \sum\limits_{k\le n} k^m p_{nk}$.

In [**3**] we introduced a concept of the generalised moments $E(K_\rho(X_n))$, where $K_\rho(x) := x^\rho \ell(x),\ x > 0;\ K_\rho(0) := 0$, is a regularly varying function of index $\rho \in \mathbb{R}$.

Thus

$$E\big(K_\rho(X_n)\big) := \sum_{k\le n} k^\rho \ell(k) p_{nk}, \quad \rho \in \mathbb{R}^+.$$

In [**3**] and [**4**] we posed the following problem:

If $E(X_n) \to \infty$, give a characterization of probability laws with the property

$$E\big(K_\rho(X_n)\big) \sim c_\rho K_\rho\big(E(X_n)\big),\ \ a < \rho < b, \quad (n \to \infty) \tag{2}$$

where c_ρ is a constant independent of n.

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That asymptotic behavior of ρ-th moment could depend only on the first moment is not so strange as it seems. For example, an elementary inequality shows that $E(X_n^\rho) \geq \big(E(X_n)\big)^\rho$ for each $\rho \geq 1$ and $E(X_n^\rho) \leq \big(E(X_n)\big)^\rho$ for $0 < \rho < 1$.

In cited papers we proved that the relation (2) actually takes place with $c_\rho = 1$, for generalised Binomial and POISSON laws.

1. PRELIMINARIES

Thoroughout the paper we have to deal with the KARAMATA's class K_ρ of regularly varying functions or sequences.

It is well known that $f \in K_\rho$ if it could be expressed in the form $f(x) := x^\rho \ell(x)$, $\rho \in \mathbb{R}$, where ρ is the index of regular variation and $\ell(x) \in K_0$ is so-called slowly varying function i.e. positive, continuous and satisfying

$$\forall t > 0, \ \ \ell(tx) \sim \ell(x) \qquad (x \to \infty).$$

Some examples of slowly varying functions are:

$$1, \quad \log^a x, \quad \log^b(\log x), \quad \exp\left(\frac{\log x}{\log\log x}\right), \quad \exp(\log^c x); \quad a, b \in \mathbb{R}, \ 0 < c < 1.$$

The theory of regular variation is presented in [**1**] and [**2**]. We quote here some facts from [**1**]:

Lemma 1. *If* $a(x) \sim b(x) \to \infty$ $(x \to \infty)$, *then* $K_\rho\big(a(x)\big) \sim K_\rho\big(b(x)\big)$ $(x \to \infty)$.

Lemma 2. *For* $\mu > 0$ *we have* $\sup_{x \leq y} x^\mu \ell(x) \sim y^\mu \ell(y)$ $(y \to \infty)$.

Lemma 3. (KARAMATA's Theorem) *Let* f *be positive and locally bounded in* $[a, \infty)$ *and* $\sigma > -\rho$. *Then the following are equivalent*

$$\text{(i)} \quad f \in K_\rho; \qquad \text{(ii)} \quad \frac{x^\sigma f(x)}{\int\limits_a^x \frac{t^\sigma f(t)}{t}\,\mathrm{d}t} \to \sigma + \rho \qquad (x \to \infty).$$

Lemma 4. (VUILLEUMIER [**5**]) *If* $\sum\limits_{k \leq n} q_{nk} \to 1$ $(n \to \infty)$ *and there exists* $\varepsilon > 0$ *such that* $\sum\limits_{k \leq n} k^{-\varepsilon} |q_{nk}| = O(n^{-\varepsilon})$ *then* $\sum\limits_{k \leq n} \ell(k) q_{nk} \sim \ell(n)$ $(n \to \infty)$, *for each* $\ell \in K_0$.

Now, we define the Sequential Law by the following: Let $\{p_k\}_{k=1}^\infty$ be a sequence of positive real numbers and put in (1) $P\{X_n = k\} = p_{nk}$, where

$$p_{nk} := \frac{p_k}{\sum\limits_{m \leq n} p_m}, \quad k = 1, 2, \ldots, n.$$

We study the asymptotic behavior of $E\big(K_\rho(X_n)\big)$ with respect to $E(X_n)$ and show that (2) holds with $c_\rho \neq 1$.

3. THE RESULTS

Throughout the rest of the paper we suppose that $\lim_n \frac{1}{n} E(X_n)$ exists. Hence, if $\lim_n \frac{1}{n} E(X_n) = c$ it is evident that $c \in [0,1]$, and we have to deal with three cases: $0 < c < 1,\ c = 0,\ c = 1$.

Proposition 1. *For* $0 < c < 1$ *and* $\mu > -\frac{c}{1-c}$ *the following are equivalent*

$$\text{(3)} \qquad \text{(i)}\ \frac{E(X_n)}{n} \to c; \qquad \text{(ii)}\ \frac{E(X_n^\mu)}{n^\mu} \to \frac{c}{\mu + c(1-\mu)} \qquad (n \to \infty).$$

Proof. Denote $C(y) := \sum_{k \le y} p_k$. Then Abel's partial summation gives

$$\text{(4)} \qquad \sum_{k \le y} k^\sigma p_k = y^\sigma C(y) - \sigma \int_1^y t^{\sigma-1} C(t)\,\mathrm{d}t, \quad \sigma \in \mathbb{R}.$$

Putting $n = [y]$ in $\lim_n \frac{1}{n} E(X_n) = c$ and $\sigma = 1$ in (4), we obtain:

$$\int_1^y C(t)\,\mathrm{d}t \Big/ \big(yC(y)\big) \to 1 - c \qquad (y \to \infty).$$

Therefore, Lemma 3 gives $C(y) \in K_{c/(1-c)}$.

Applying again this lemma with $\rho = \frac{c}{1-c}$, $\sigma = \mu$, from (4) we get that

$$\frac{\sum_{k \le y} k^\mu p_k}{y^\mu C(y)} = 1 - \mu \frac{\int_1^y t^{\mu-1} C(t)\,\mathrm{d}t}{y^\mu C(y)} \to 1 - \mu \frac{1}{\mu + \frac{c}{1-c}} = \frac{c}{\mu + c(1-\mu)} \quad (y \to \infty),$$

which gives the right-hand side of (3) with $y = n$.

Conversely, supposing the validity of

$$\frac{1}{n^\mu} E(X_n^\mu) \to \frac{c}{\mu + c(1-\mu)} \qquad (n \to \infty)$$

with $n = [y]$, then (4) gives

$$y^\mu C(y) \Big/ \int_1^y t^{\mu-1} C(t)\,\mathrm{d}t \to \mu + \frac{c}{1-c} \qquad (y \to \infty)$$

i.e. by Lemma 3, $C(y) \in K_{c/(1-c)}$. Using again Lemma 3. with $\sigma = 1,\ f := C$ we obtain the left-hand side of (3).

The case $c = 1$ needs a different approach.

Proposition 2. *For each $\mu \in \mathbb{R}$, the following are equivalent*

$$\text{(i)} \quad E(X_n) \sim n; \qquad \text{(ii)} \quad E(X_n^{\mu}) \sim n^{\mu} \qquad (n \to \infty).$$

Proof. The condition $\dfrac{1}{n} E(X_n) \to 1 \ \ (n \to \infty)$ implies

$$\int_1^y C(t)\,\mathrm{d}t \Big/ \big(yC(y)\big) \to 0 \qquad (y \to \infty). \tag{5}$$

Denote by A the class of functions $C(y)$ satisfying (5). We have

Proposition 2.1. *The following are equivalent*

$$\text{(i)} \quad C(y) \in A; \qquad \text{(ii)} \quad y^{\sigma} C(y) \in A,$$

for each $\sigma \in \mathbb{R}$.

For the proof we need two lemmas below.

Lemma 2.1. *If $C(y) \in A$ then $y^{\sigma} C(y) \to \infty$, $(y \to \infty)$ for any fixed $\sigma \in \mathbb{R}$.*

Proof. Since $yC(y) \Big/ \int_1^y C(t)\,\mathrm{d}t \to \infty \ \ (y \to \infty)$, we can find $y_0 > 1$ such that

$$yC(y) \Big/ \int_1^y C(t)\,\mathrm{d}t > |\sigma| + 2, \quad y > y_0, \tag{6}$$

i.e.

$$D\left(\log \int_1^y C(t)\,\mathrm{d}t\right) > \frac{|\sigma| + 2}{y}, \qquad y > y_0. \tag{7}$$

Integrating (7) over $[y_0, y]$, we obtain

$$\int_1^y C(t)\,\mathrm{d}t > c\big(y_0, |\sigma|\big) y^{|\sigma|+2}, \quad y > y_0$$

i.e. taking in account (6),

$$y^{\sigma} C(y) > c'\big(y_0, |\sigma|\big) y^{\sigma + |\sigma| + 1}, \ \ y > y_0,$$

and the assertion follows.

Lemma 2.2. *If $C(y) \in A$, then* $\int_1^y t^{\sigma} C(t)\,\mathrm{d}t \sim y^{\sigma} \int_1^y C(t)\,\mathrm{d}t \ \ (y \to \infty)$ *for any fixed $\sigma \in \mathbb{R}$.*

Proof. According to Lemma 2.1 and (5), we have

$$\frac{\int_1^y t^{\sigma} C(t)\,\mathrm{d}t}{y^{\sigma} \int_1^y C(t)\,\mathrm{d}t} \to \frac{\mathrm{D}\left(\int_1^y t^{\sigma} C(t)\,\mathrm{d}t\right)}{\mathrm{D}\left(y^{\sigma} \int_1^y C(t)\,\mathrm{d}t\right)} = \frac{1}{1 + \sigma \dfrac{\int_1^y C(t)\,\mathrm{d}t}{yC(y)}} \to 1 \quad (y \to \infty),$$

where D denotes $\mathrm{d}/\mathrm{d}y$.

Proof of Proposition 2.1.

If $C(y) \in A$ then, according to Lemma 2.2,

$$\frac{\int_1^y t^\sigma C(t)\,\mathrm{d}t}{y^{\sigma+1}C(y)} = \frac{\int_1^y C(t)\,\mathrm{d}t}{yC(y)} \, \frac{\int_1^y t^\sigma C(t)\,\mathrm{d}t}{y^\sigma \int_1^y C(t)\,\mathrm{d}t} \to 0 \quad (y \to \infty),$$

i.e. $y^\sigma C(y) \in A$, $\forall \sigma \in \mathbb{R}$.

Conversely, if for some $\sigma \in \mathbb{R}$, $y^\sigma C(y) \in A$ then

$$\frac{\int_1^y C(t)\,\mathrm{d}t}{yC(y)} = \frac{\int_1^y t^{-\sigma}\big(t^\sigma C(t)\,\mathrm{d}t\big)}{yC(y)} \sim \frac{y^{-\sigma}\int_1^y t^\sigma C(t)\,\mathrm{d}t}{yC(y)} = \frac{\int_1^y t^\sigma C(t)\,\mathrm{d}t}{y^{\sigma+1}C(y)} \to 0 \quad (y \to \infty),$$

i.e. $C(y) \in A$.

Using Proposition 2.1 and the fact

$$\frac{\sum_{k \le y} k^\sigma p_k}{y^\sigma C(y)} = 1 - \sigma \, \frac{\int_1^y t^{\sigma-1} C(t)\,\mathrm{d}t}{y^\sigma C(y)}$$

with $\sigma = 1$ and $\sigma = \mu$, the proof of the Proposition 2 readily follows. □

Propositions 1 and 2 are basic for the estimation of $E\big(K_\mu(X_n)\big)$. Namely, putting in Lemma 4,

$$q_{nk} := \frac{\mu + c(1-\mu)}{cC(n)} \left(\frac{k}{n}\right)^\mu p_k, \; 0 < c < 1, \; \delta > 0, \; \mu > \delta - \frac{c}{1-c}\,,$$

Proposition 1 gives $\sum_{k \le n} q_{nk} \to 1 \quad (n \to \infty)$, and the condition from Lemma 4 is satisfied with $\varepsilon = \delta/2$. Thus we have

Proposition 3. *For* $0 < c < 1, \; \delta > 0, \; \mu > \delta - \frac{c}{1-c}\,, \; E(X_n) \sim cn \quad (n \to \infty)$, *implies*

$$E\big(X_n^\mu \ell(X_n)\big) \sim \frac{c}{\mu + c(1-\mu)}\, n^\mu \ell(n) \qquad (n \to \infty).$$

In the same way, using the Proposition 2, we get

Proposition 4. *If* $E(X_n) \sim n \;\; (n \to \infty)$, *then*

$$E\big(X_n^\mu \ell(X_n)\big) \sim n^\mu \ell(n) \qquad (n \to \infty),$$

for any $\mu \in \mathbb{R}$.

In the case $c = 0$, Lemma 3 gives $C(y) \in K_0$ and we have

$$\forall \mu > 0 : E(X_n^\mu) = o(n^\mu) \quad (n \to \infty). \tag{8}$$

Hence,

Proposition 5. *If* $\frac{1}{n} E(X_n) \to 0 \ (n \to \infty)$, *then*

$$E\big(X_n^\mu \ell(X_n)\big) = o\big(n^\mu \ell(n)\big) \quad (n \to \infty),$$

for any $\mu \in \mathbb{R}^+$, $\ell \in K_0$.

Proof. Applying Lemma 2 and (8), we get

$$\frac{E\big(X_n^\mu \ell(X_n)\big)}{n^\mu \ell(n)} = O\Big(\sup_{k \le n} \big(k^{\mu/2} \ell(k)\big)\Big) \frac{E(X_n^{\mu/2})}{n^\mu \ell(n)}$$

$$= O\big(n^{\mu/2} \ell(n)\big) \frac{o(n^{\mu/2})}{n^\mu \ell(n)} = o(1) \quad (n \to \infty).$$

It is evident that, summarising results from Propositions 3, 4 and 5, for the generalised moments of positive order we can formulate the following

Proposition 6. *If* $\frac{1}{n} E(X_n) \to c, \ (n \to \infty), \ c > 0$, *then*

$$E\big(X_n^\mu \ell(X_n)\big) \sim \frac{c^{1-\mu}}{\mu + c(1-\mu)} \big(E(X_n)\big)^\mu \ell\big(E(X_n)\big), \ \mu \in \mathbb{R}^+, \ \ell \in K_0 \quad (n \to \infty);$$

therefore giving an answer to the question posed in (2) in the case of the sequential law.

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