

## ON A CLASS OF TRICYCLIC REFLEXIVE CACTUSES

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A simple graph is reflexive if the second largest eigenvalue of its  $(0, 1)$  adjacency matrix does not exceed 2. A graph is a cactus, or a treelike graph, if any pair of its cycles (circuits) has at most one common vertex. The subject of this paper is the class of tricyclic cactuses in which the central cycle is a quadrangle touching the rest two cycles at its non-adjacent vertices. In this class we describe a set of maximal reflexive graphs. The so-called “pouring” of SMITH trees plays the crucial role in characterizing the resulting set.

### 1. INTRODUCTION

For a non-oriented graph  $G$  without loops or multiple edges (shortly a simple graph),  $P_G(\lambda) = \det(\lambda I - A)$ , i.e. the characteristic polynomial of its  $(0, 1)$  adjacency matrix  $A$ , is called the *characteristic polynomial* of  $G$ , and we will denote it by  $P(\lambda)$  if it is clear which graph it is related to. The family of its roots (the *eigenvalues* of  $G$ ) is the *spectrum* of  $G$ . For a simple graph it consists of real numbers and we will denote them in their non-increasing order:  $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$ . In a connected graph the largest eigenvalue  $\lambda_1$  (also called the *index* of the graph) is strictly greater than  $\lambda_2$ , while for a disconnected graph, since now the spectrum is the union of the spectra of its components,  $\lambda_1$  equals  $\lambda_2$  if this is the common index of two distinct components. The so-called *interlacing theorem* establishes the interrelation between the spectra of a graph and its induced subgraphs.

*Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  be the eigenvalues of a graph  $G$  and  $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$  eigenvalues of its induced subgraph  $H$ . Then the inequalities  $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$  hold.*

Thus e.g. if  $m = n - 1$ , it will be  $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots$ , and also  $\lambda_1 > \mu_1$  if  $G$  is connected.

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Graphs having  $\lambda_2 \leq 2$  are usually called *reflexive graphs*, and if  $\lambda_2 \leq 2 \leq \lambda_1$  they are often called *hyperbolic graphs*. Reflexive graphs correspond to some sets of vectors in the LORENTZ space  $R^{p,1}$  and are interesting since they have some application to the construction and classification of reflexion groups [7]. So far, reflexive trees have been considered in [5] and [6] and a class of bicyclic reflexive graphs in [12] (see also [8]). Recently, multicyclic reflexive cactuses have been studied in [9], [10] and [11].

A *cactus*, or a *treelike graph*, is a graph whose any two cycles have at most one common vertex (i.e. are edge-disjoint).

Since the interlacing theorem implies that the spectral property  $\lambda_2 \leq 2$  is *hereditary* (every induced subgraph preserves it), the results of this paper, as well as of some former ones, will be expressed through sets of maximal graphs having this property. Also, since a graph with such a spectral property can have arbitrary number of components, it goes without saying that we consider only connected graphs.

The terminology of the theory of graph spectra in this paper follows [1], while for other graph theoretic notions one can see [4].

## 2. SOME FORMER AND AUXILIARY RESULTS

Graphs whose index is equal to 2 are described in the following well known Lemma:

**Lemma 1.** ([14], see also [1], p. 79) *The index of a graph  $G$  satisfies  $\lambda_1(G) \leq 2$  ( $\lambda_1(G) < 2$ ) if and only if each component of  $G$  is a subgraph (a proper subgraph) of some of the graphs displayed in Fig. 1, all of which have  $\lambda_1 = 2$ .*

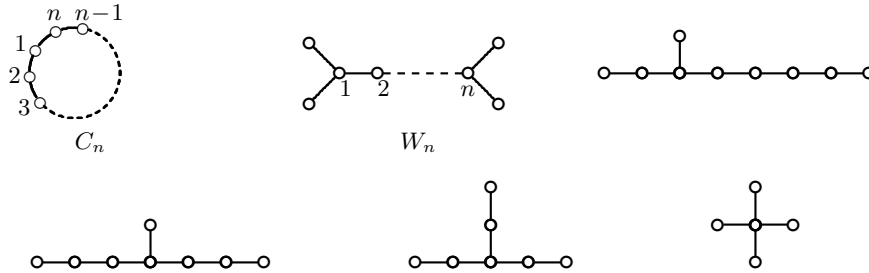


Figure 1.

These graphs are known as *Smith graphs*.

One of the suitable tools for calculating values  $P_G(2)$  of treelike graphs is given by the following lemma, due to A. SCHWENK.

**Lemma 2.** ([13], see also [1], p. 78) *Given a graph  $G$ , let  $C(v)$  denote the set of all*

*cycles containing a vertex  $v$ . Then*

$$P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum_{u \in \text{adj}(v)} P_{G-v-u}(\lambda) - 2 \sum_{C \in \mathcal{C}(v)} P_{G-V(C)}(\lambda),$$

where  $\text{Adj}(v)$  denotes the set of neighbours of  $v$  and  $G-V(C)$  is the graph obtained from  $G$  by removing the vertices belonging to the cycle  $C$ .

**Corollary 1.** (E. HEILBRONNER - see e.g. [1], p. 59) *Let  $G$  be a graph with a pendant edge  $v_1v_2$  ( $v_1$  being of degree 1). Then*

$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - P_{G_2}(\lambda),$$

where  $G_1$  ( $G_2$ ) is the graph obtained from  $G$  (resp.  $G_1$ ) by deleting vertex  $v_1$  ( $v_2$ ).

First supergraphs of the SMITH graphs have the following property.

**Lemma 3.** [12] *Let  $G$  be a connected graph obtained by extending any of the Smith graphs (Fig. 1) by a vertex of arbitrary degree. Then  $P_G(2) < 0$  (i.e.  $\lambda_2(G) < 2 < \lambda_1(G)$ ).*

For some types of graphs, values  $P_G(2)$  can easily be calculated. Let  $P_n$  denote the path with  $n$  vertices ( $n - 1$  edges).

**Lemma 4.** [12]  $P_{P_n}(2) = n + 2$ .

For a lot of treelike graphs the property  $\lambda_2 \leq 2$  can be tested in a very simple way, namely by identifying and removing a single cut-vertex. If such a removal decomposes it into two components which are both Smith graphs, according to the interlacing theorem we get  $\lambda_2(G) = 2$ . A generalization of this fact is given by the following theorem.

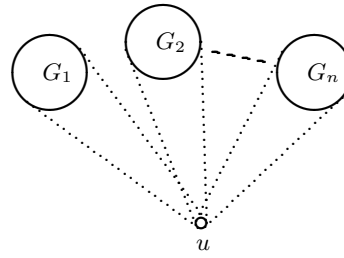


Figure 2.

**Theorem 1.** [12] *Let  $G$  be a graph of the form displayed in Fig. 2,  $u$  being a cut-vertex.*

1° *If at least two components of  $G - u$  are supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then  $\lambda_2(G) > 2$ .*

2° *If at least two components of  $G - u$  are Smith graphs, and the rest are subgraphs of Smith graphs, then  $\lambda_2(G) = 2$ .*

3° *If at most one component of  $G - u$  is a Smith graph, and the rest are proper subgraphs of Smith graphs, then  $\lambda_2(G) < 2$ .*

Of course, there are also a lot of cases not to be covered by this theorem (one proper supergraph and the rest of proper subgraphs of SMITH graphs). Such cases are the subject of further investigations, and therefore we always presuppose that Theorem 1 is not applicable.

If all cycles of a cactus have the unique common vertex we say that they form a *bundle*. Since in this case the problem of finding all maximal reflexive cactuses is harder than otherwise, it should be treated separately.

All maximal reflexive bicyclic graphs with the bridge between the cycles have been found in [12]. An interesting part of the result, which is important for the result of this paper, is contained in the following theorem. Let two cycles of arbitrary lengths be joined by a bridge whose vertices are  $c_1$  and  $c_2$ , and let  $c_1c_3$  be additional pendent edge.

**Theorem 2.** ([12], for a variant proof see also [11]) *If in a bicyclic graph with a bridge between its cycles all vertices of the cycles except  $c_i$  ( $i = 1, 2$ ) are of degree two and if Theorem 1 is not applicable, it is reflexive if and only if it is an induced subgraph of a graph formed by identifying with  $c_2$  and  $c_3$  two vertices obtained by splitting any of the Smith trees  $S$  at any vertex into two trees  $S_1$  and  $S_2$  (Fig. 3(a)), or of the graph of Fig. 3 (b) for  $\ell_1 = \ell_2 = 0$ .*

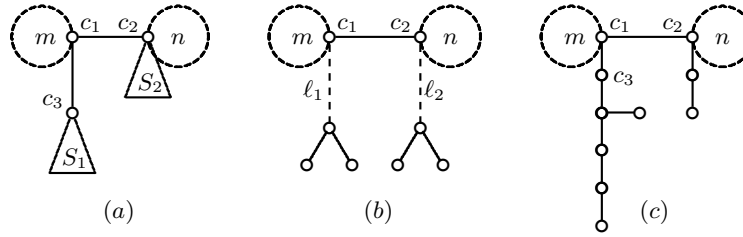


Figure 3.

Obviously, if  $\max(\ell_1, \ell_2) \geq 1$ , the graph of the case (b) fits in (a).

The possibility of splitting a given SMITH tree at any vertex and leaning its parts  $S_1$  and  $S_2$  on  $c_2$  and  $c_3$  (e.g. as in Fig. 3(c)) produces an interesting phenomenon, which we will call *pouring* of SMITH trees (between the vertices  $c_2$  and  $c_3$ ).

Naturally, this description includes also leaning of a whole SMITH tree on  $c_2$ , while  $c_3$  remains an end-vertex. But since a cycle is a SMITH graph too, we finally find out that the tricyclic graph  $T_0$  of Fig. 4 is also a maximal reflexive graph, being the only maximal reflexive cactus with more than two cycles possessing a bridge between its cycles and to which Theorem 1 cannot be applied [12].

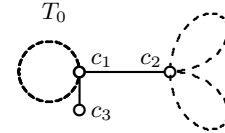


Figure 4.

Notice that leaning a whole SMITH graph at  $c_3$  produces a situation covered by Theorem 1.

Reflexive treelike graphs with more than three cycles have been considered in [10] and [11].

**Theorem 3.** ([10], [11]) *A reflexive cactus to which Theorem 1 cannot be applied and whose cycles do not make a bundle has at most five cycles. The only such graphs with five cycles, which are all maximal, are the four families of graphs  $Q_1$ ,  $Q_2$ ,  $T_1$  and  $T_2$  displayed in Fig. 5.*

The case of four cycles has been solved in [10] and [11].The solution (set of

all maximal reflexive graphs in this class) includes also several cases of pouring of SMITH trees.

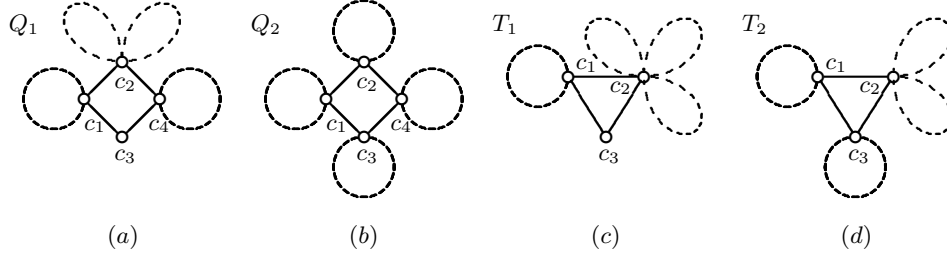


Figure 5.

### 3. THE CASE OF THREE CYCLES

Suppose that the cycles of a tricyclic reflexive cactus do not form a bundle. Then one of them is the *central* cycle (it has touching vertices with the rest two ones, which will be called *outer* cycles). If the two touching vertices of the central cycle are not adjacent, it must be a quadrangle (otherwise, Theorem 1 says that such a cactus is not reflexive). If these vertices are adjacent and if the central cycle is triangle or quadrangle, according to Theorem 3 such cases allow adding of new cycles, which then can be replaced by SMITH trees ([10], [11]), and they are to be considered separately. If the central cycle is at least pentagon, such a reflexive cactus cannot have more than three cycles. These are the four characteristic cases of tricyclic reflexive cactuses (Fig. 6).

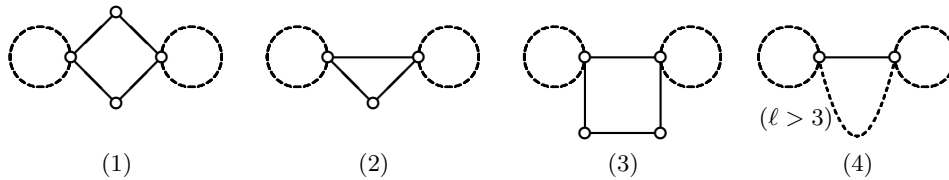


Figure 6.

The result of this paper gives a class of maximal reflexive tricyclic cactuses within the scope of the case (1).

### 4. POURING OF A PAIR OF SMITH TREES AND A CLASS OF MAXIMAL REFLEXIVE TRICYCLIC CACTUSES

The family of graphs of Fig. 3 is a set of maximal reflexive cactuses within the class of bicyclic graphs with a bridge between the cycles. This set can also be described in the following way: starting from the graph  $T_0$  of Fig. 4, let us replace one of the cycles at the vertex  $c_2$  by any of SMITH trees, and let us pour it between  $c_2$  and  $c_3$ .

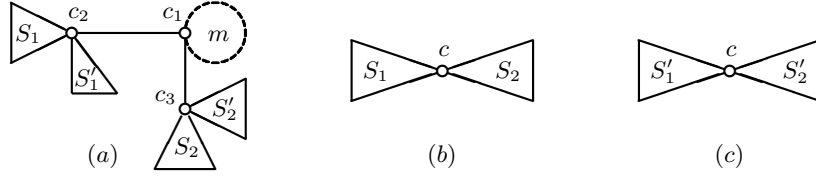


Figure 7.

But the natural next step is to do the same with both cycles at  $c_2$ .

**Lemma 5.** *Let the unicyclic cactus  $G$  of Fig. 7(a) ( $m$  is the length of the cycle) be such that the trees of Fig. 7(b) and (c) (obtained by identifying  $c$ -vertices of  $S_1$  and  $S_2$ , and also  $S'_1$  and  $S'_2$ ) are Smith trees. Then  $\lambda_2(G) = 2$ , and every extension of  $G$  by a vertex joined to any vertex of  $S_1$ ,  $S'_1$ ,  $S_2$  or  $S'_2$  implies  $\lambda_2 > 2$ .*

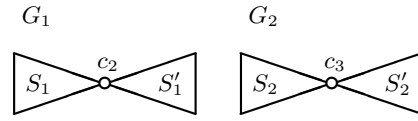


Figure 8.

**Proof.** Applying Lemma 2 to the graph  $G_1$  of Fig. 8 (with respect to the vertex  $c_2$ ) we get

$$(1) \quad P_{G_1}(2) = 2P_{S_1-c_2}(2)P_{S'_1-c_2}(2) - P_{S'_1-c_2}(2) \sum_{d \in S_1 \cap \text{adj } c_2} P_{S_1-c_2-d}(2) \\ - P_{S_1-c_2}(2) \sum_{d \in S'_1 \cap \text{adj } c_2} P_{S'_1-c_2-d}(2)$$

(where  $\text{adj } c_2$  means the set of vertices adjacent to  $c_2$ ), and the corresponding expression for  $P_{G_2}(2)$ . For the sake of brevity let us introduce

$$\sum_{d \in S_1 \cap \text{adj } c_2} P_{S_1-c_2-d}(2) = \sum_1, \quad \sum_{d \in S'_1 \cap \text{adj } c_2} P_{S'_1-c_2-d}(2) = \sum'_1,$$

and the corresponding  $\sum_2$  and  $\sum'_2$  in  $P_{G_2}(2)$ . Now, let us apply Lemma 2 to the whole graph  $G$  with respect to  $c_1$ . Putting  $P_{S_i-c_i}(2) = P_i$ ,  $P_{S'_i-c_i}(2) = P'_i$ , ( $i = 1, 2$ ) and applying (1) and Lemma 4 we find

$$(2) \quad P_G(2) = 2mP_{G_1}(2)P_{G_2}(2) - 2(m-1)P_{G_1}(2)P_{G_2}(2) - mP_{G_1}(2)P_{G_2-c_3}(2) \\ - mP_{G_2}(2)P_{G_1-c_2}(2) - 2P_{G_1}(2)P_{G_2}(2) \\ = -m \left( (2P_1P'_1 - P'_1\sum_1 - P_1\sum'_1)P_2P'_2 + (2P_2P'_2 - P'_2\sum_2 - P_2\sum'_2)P_1P'_1 \right) \\ = m \left( -4P_1P'_1P_2P'_2 + P_1P'_1P'_2\sum_2 + P_1P'_1P_2\sum'_2 + P'_1P_2P'_2\sum_1 + P_1P_2P'_2\sum'_1 \right).$$

But since the graphs of Fig. 7 (b) and (c) are SMITH graphs, we get

$$(3) \quad P(2) = 2P_1P_2 - P_1\sum_2 - P_2\sum_1 = 0$$

implying

$$(4) \quad P_1 \sum_2 + P_2 \sum_1 = 2P_1 P_2,$$

and analogously

$$(3') \quad P(2) = 2P'_1 P'_2 - P'_1 \sum'_2 - P'_2 \sum'_1 = 0,$$

$$(4') \quad P'_1 \sum'_2 + P'_2 \sum'_1 = 2P'_1 P'_2.$$

Application of (4) and (4') to (2) gives  $P_G(2) = 0$ .

On the other hand, if at least one of the graphs of Fig. 7 (b) and (c) is a SMITH tree extended by one vertex (joined to an arbitrary one), then by Lemma 3, in the relations (3) and (3') we have  $P(2) < 0$ , which changes (4) and (4') into inequalities. Applying them to (2) we get  $P_G(2) > 0$ . Analogously, if at least one of those two graphs is a proper subgraph of a SMITH tree, then  $P_G(2) < 0$ . Since  $P_G(\lambda) = \det(\lambda I - A) > 0$  for  $\lambda > \lambda_1$  and  $P_G(\lambda) < 0$  for  $\lambda_2 < \lambda < \lambda_1$ , we see that  $P_G(2) = 0$  implies  $\lambda_2(G) = 2$  and that  $\lambda_2(G) > 2$  ( $\lambda_2(G) < 2$ ) if  $P_G(2) > 0$  ( $P_G(2) < 0$  resp.). This completes the proof of the Lemma.  $\square$

This result enables describing a family of maximal tricyclic reflexive cactuses.

**Theorem 4.** (The main result) *Let a tricyclic cactus have the cyclic structure as that of Fig. 6(1), let  $S_1, S'_1, S_2$  and  $S'_2$  be parts of two Smith trees as described in Lemma 5 (Fig. 7), and let them be leaned on the vertices  $c_2$  and  $c_3$  of the quadrangle as shown at Fig. 9. Then all graphs of this set are maximal reflexive graphs.*

**Proof.** If we join two arbitrary cycles by the path  $c_1 c_2 c_4$  of length 2, by Theorem 1 we find  $\lambda_2 = 2$ . If we introduce a new vertex  $c_3$  and join it to  $c_2$  and  $c_4$ , by Lemma 2 again  $P(2) = 0$  and  $\lambda_2 = 2$ . It follows now by Corollary 1 and induction that arbitrary extension of this graph by trees leaned on  $c_2$  and  $c_3$  preserves the property  $P(2) = 0$  and we should find out which of such graphs have just  $\lambda_2 = 2$ . But one can make sure that all graphs of Fig. 9 have  $\lambda_2 = 2$ . On the other hand, these graphs cannot be extended at any vertex of its outer cycles, including  $c_1$  and  $c_4$ , since then, after removing e.g.  $c_3$  and applying Theorem 1 to  $c_2$ , we find  $\lambda_2 > 2$ . Also, they cannot be extended at any other vertex, because if we do it by a single new vertex, according to Lemma 5 we see that a subgraph of it has  $\lambda_2 > 2$ . Thus, all graphs of Fig. 9 are maximal reflexive cactuses and the proof is complete.  $\square$

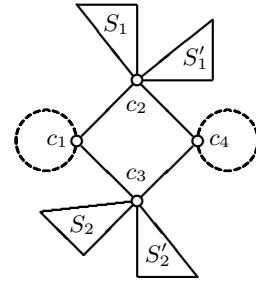


Figure 9.

Of course, the result includes the possibility of leaning a whole SMITH tree on  $c_2$  and another SMITH tree on  $c_3$ , or leaning two SMITH trees on one of these two vertices. This is the obvious first step suggested by the graphs  $Q_1$  and  $Q_2$  of Theorem 3, and thus Theorem 4 appears as a natural generalization of Theorem

3 and the corresponding results concerning maximal reflexive cactuses with four cycles [11] (instead of two intact SMITH trees, we have their pouring between  $c_2$  and  $c_3$ ).

It should be noted here that this family does not include all maximal reflexive graphs generated by the graph of Fig. 6 (1). In the variety of assumptions for the trees leaned on  $c_2$  and  $c_3$  we can recognize cases which fit in Theorem 4 and those which have to be investigated in some other way. In fact, whenever Theorem 4 cannot be applied, we have the situation that a proper supergraph  $S$  of a SMITH tree is leaned e. g. on  $c_2$  (without or together with some additional parts also leaned on  $c_2$ ), and that  $S$  is such a tree that after removing  $c_2$  it becomes a proper subgraph of a SMITH tree (since otherwise Theorem 1 could be applied). The process of finding these remaining maximal reflexive graphs necessitates a careful discussion of particular cases and it will be the subject of some subsequent paper.

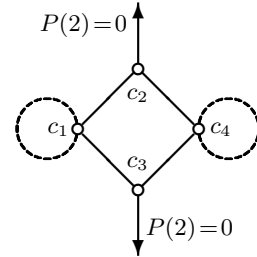


Figure 10.

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