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NOTE ON IYENGAR'S INEQUALITY

Zheng Liu

The classical Iyengar's inequality and its generalization are recaptured on certain weaker conditions. A related Iyengar's type integral inequality and its generalization are also considered.

1. INTRODUCTION

The following inequality was established by K. S. K. Iyengar in 1938 by means of geometrical consideration for functions whose firste derivative is bounded as follows:

Theorem A. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that for all $x \in [a,b]$ with $M > 0$ we have $|f'(x)| \le M$. Then*

$$(1) \qquad \left| \int_a^b f(x)\,\mathrm{d}x - \frac{f(a)+f(b)}{2}\,(b-a) \right| \le \frac{M(b-a)^2}{4} - \frac{\big(f(b)-f(a)\big)^2}{4M}\,.$$

In 1996, Agarwal and Dragomir [**2**] applied Hayashi's inequality to obtain inequality which is generalization of Iyengar inequality (1) as:

Theorem B. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that for all $x \in [a,b]$ with $M > m$ we have $m \le f'(x) \le M$. Then*

$$(2) \qquad \left| \int_a^b f(x)\,\mathrm{d}x - \frac{f(a)+f(b)}{2}\,(b-a) \right| \le \frac{\big(f(b)-f(a)-m(b-a)\big)\big(M(b-a)-f(b)+f(a)\big)}{2(M-m)}\,.$$

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It should be noted that Theorem B and Theorem A are equivalent, in the sense that we can also obtain Theorem B from Theorem A. Indeed, we can write the condition $m \leq f'(x) \leq M$ as $\left| f'(x) - \frac{m+M}{2} \right| \leq \frac{M-m}{2}$. So, let $g(x) = f(x) - \frac{m+M}{2}\, x$ and $M_1 = \frac{M-m}{2}$, if we apply Theorem A on g, i.e., using the inequality (1) for g and M_1, we shall obtain the inequality (2).

In 1988, Elezović and Pečarić obtained the inequality (2) under weaker condition on function f by using the Hayashi's inequality as follows:

Theorem C. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that for all $x \in [a,b]$ with $M > m$ we have*

$$m \leq \frac{f(x) - f(a)}{x-a} \leq M \quad \text{and} \quad m \leq \frac{f(b) - f(x)}{b-x} \leq M.$$

If f' is integrable on $[a,b]$, then the inequality (2) holds.

In [**4**], Qi has cited and deduced a more related Iyengar type integral inequality involving boundend second-order derivative as:

Theorem D. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable function such that for all $x \in [a,b]$ with $M > 0$ we have $|f''(x)| \leq M$. Then*

$$\left| \int_a^b f(x)\,\mathrm{d}x - \frac{f(a)+f(b)}{2}\,(b-a) + \frac{1+Q^2}{8}\,\big(f'(b) - f'(a)\big)(b-a)^2 \right| < \frac{M(b-a)^3}{24}\,(1 - 3Q^2), \tag{3}$$

where

$$Q^2 = \frac{\left(f'(a) + f'(b) - 2\,\dfrac{f(b)-f(a)}{b-a} \right)^2}{M^2(b-a)^2 - \big(f'(b) - f'(a)\big)^2}\,. \tag{4}$$

Here we have given revised version for (4) since the expression in [**4**] as well as in [**5**] and [**6**] contained a misprint.

In this paper, the inequalities (1), (2) and (3) will be recaptured on certain weaker conditions and a generalization of the inequality (3) is given.

ON INEQUALITIES (1) AND (2)

We first consider inequalities (1) and (2) for functions that are not necessarily differentiable.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function such that for all $x \in [a,b]$ with $M > 0$ we have*

$$|f(x) - f(a)| \leq M(x-a) \quad \text{and} \quad |f(x) - f(b)| \leq M(b-x). \tag{5}$$

Then the inequality (1) *holds.*

Proof. By (5), it is clear that for all $x \in [a,b]$ we have

$$f(a) - M(x-a) \le f(x) \le f(a) + M(x-a)$$

and

$$f(b) - M(b-x) \le f(x) \le f(b) + M(b-x).$$

For any $t \in [a,b]$, it is immediate that

$$f(a)(t-a) - \frac{M}{2}(t-a)^2 \le \int_a^t f(x)\,\mathrm{d}x \le f(a)(t-a) + \frac{M}{2}(t-a)^2$$

and

$$f(b)(b-t) - \frac{M}{2}(b-t)^2 \le \int_t^b f(x)\,\mathrm{d}x \le f(b)(b-t) + \frac{M}{2}(b-t)^2.$$

Then we have

$$\begin{aligned} (6)\qquad f(a)(t-a) + f(b)(b-t) - \frac{M}{2}\big((t-a)^2 + (b-t)^2\big) &\le \int_a^b f(x)\,\mathrm{d}x \\ &\le f(a)(t-a) + f(b)(b-t) + \frac{M}{2}\big((t-a)^2 + (b-t)^2\big). \end{aligned}$$

It is not difficult to find that the values of the left-hand side of (6) reach a maximum at

$$t_1 = \frac{a+b}{2} - \frac{f(b)-f(a)}{2M} \in [a,b]$$

and the right-hand side of (6) reaches a minimum at

$$t_2 = \frac{a+b}{2} + \frac{f(b)-f(a)}{2M} \in [a,b]$$

respectively. Thus we can deduce that

$$\begin{aligned} \frac{f(a)+f(b)}{2}(b-a) - \frac{M(b-a)^2}{4} + \frac{\big(f(b)-f(a)\big)^2}{4M} &\le \int_a^b f(x)\,\mathrm{d}x \\ &\le \frac{f(a)+f(b)}{2}(b-a) + \frac{M(b-a)^2}{4} - \frac{\big(f(b)-f(a)\big)^2}{4M}. \end{aligned}$$

Consequently, the inequality (1) follows.

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function such that for all $x \in [a,b]$ with $M > m$ we have*

$$(7)\qquad m \le \frac{f(x)-f(a)}{x-a} \le M \quad \text{and} \quad m \le \frac{f(b)-f(x)}{b-x} \le M.$$

Then the inequality (2) *holds.*

Proof. It is clear that condition (7) can be given as

$$|h(x) - h(a)| \le M_1(x-a) \quad \text{and} \quad |h(x) - h(b)| \le M_1(b-x),$$

where $h(x) = f(x) - \dfrac{M+m}{2}\, x$ and $M_1 = \dfrac{M-m}{2}$. So if we apply Theorem 1 on h, i.e., using the inequality (1) for h and M_1, we shall obtain the inequality (2).

ON THE INEQUALITY (3)

Motivated by [**7**], we now consider the inequality (3) under weaker assumption of functions that are not necessarily twice differentiable.

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function such that f' is integrable on $[a,b]$ and for all $x \in [a,b]$ with $M > 0$ we have*

$$|f'(x) - f'(a)| \le M(x-a) \quad \text{and} \quad |f'(x) - f'(b)| \le M(b-x). \tag{8}$$

Then the inequality (3) *holds.*

Proof. By (8), for all $x \in [a,b]$ we have

$$f(x) - f(a) - f'(a)(x-a) = \int_a^x \big(f'(u) - f'(a)\big)\,\mathrm{d}u \le \frac{M}{2}\,(x-a)^2,$$
$$f(x) - f(b) + f'(b)(b-x) = \int_x^b \big(-f'(u) + f'(b)\big)\,\mathrm{d}u \le \frac{M}{2}\,(b-x)^2,$$

and

$$f(x) - f(a) - f'(a)(x-a) = \int_a^x \big(f'(u) - f'(a)\big)\,\mathrm{d}u \ge -\frac{M}{2}\,(x-a)^2,$$
$$f(x) - f(b) + f'(b)(b-x) = \int_x^b \big(-f'(u) + f'(b)\big)\,\mathrm{d}u \ge -\frac{M}{2}\,(b-x)^2.$$

These imply that

$$f(a) + f'(a)(x-a) - \frac{M}{2}\,(x-a)^2 \le f(x) \le f(a) + f'(a)(x-a) + \frac{M}{2}\,(x-a)^2$$

and

$$f(b) - f'(b)(b-x) - \frac{M}{2}\,(b-x)^2 \le f(x) \le f(b) - f'(b)(b-x) + \frac{M}{2}\,(b-x)^2.$$

So for any $t \in [a,b]$ we obtain

$$f(a)(t-a) + \frac{f'(a)}{2}\,(t-a)^2 - \frac{M}{6}\,(t-a)^3 \le \int_a^t f(x)\,\mathrm{d}x$$
$$\le f(a)(t-a) + \frac{f'(a)}{2}\,(t-a)^2 + \frac{M}{6}\,(t-a)^3$$

and

$$f(b)(b-t)-\frac{f'(b)}{2}(b-t)^2-\frac{M}{6}(b-t)^3 \le \int_t^b f(x)\,\mathrm{d}x \le f(b)(b-t)-\frac{f'(b)}{2}(b-t)^2+\frac{M}{6}(b-t)^3.$$

Hence

$$\begin{aligned}(9)\qquad & f(a)(t-a)+f(b)(b-t)+\frac{f'(a)}{2}(t-a)^2-\frac{f'(b)}{2}(b-t)^2\\ &-\frac{M}{6}\big((t-a)^3+(b-t)^3\big) \le \int_a^b f(x)\,\mathrm{d}x \le f(a)(t-a)+f(b)(b-t)\\ &+\frac{f'(a)}{2}(t-a)^2-\frac{f'(b)}{2}(b-t)^2+\frac{M}{6}\big((t-a)^3+(b-t)^3\big).\end{aligned}$$

It is not difficult to find that the value of the left-hand side of (9) takes a maximum at

$$t_3=\frac{a+b}{2}+\frac{b-a}{2}\cdot\frac{f'(a)+f'(b)-2\,\dfrac{f(b)-f(a)}{b-a}}{M(b-a)+f'(b)-f'(a)} \in [a,b]$$

and the value od the right-hand side of (9) takes a minimum at

$$t_4=\frac{a+b}{2}-\frac{b-a}{2}\cdot\frac{f'(a)+f'(b)-2\,\dfrac{f(b)-f(a)}{b-a}}{M(b-a)-f'(b)+f'(a)} \in [a,b]$$

respectively. Thus we can deduce that

$$\begin{aligned}&-\frac{M(b-a)^3}{24}+\frac{(b-a)^2}{8}\cdot\frac{\left(f'(a)+f'(b)-2\,\dfrac{f(b)-f(a)}{b-a}\right)^2}{M(b-a)+f'(b)-f'(a)}\\ &\qquad\le \int_a^b f(x)\,\mathrm{d}x-\frac{f(a)+f(b)}{2}(b-a)+\frac{f'(b)-f'(a)}{8}(b-a)^2\\ &\qquad\le \frac{M(b-a)^3}{24}-\frac{(b-a)^2}{8}\cdot\frac{\left(f'(a)+f'(b)-2\,\dfrac{f(b)-f(a)}{b-a}\right)^2}{M(b-a)-f'(b)+f'(a)},\end{aligned}$$

i.e.,

$$\begin{aligned}&-\frac{M(b-a)^3}{24}+\frac{M(b-a)^3}{8}Q^2-\frac{f'(b)-f'(a)}{8}(b-a)^2Q^2\\ &\qquad\le \int_a^b f(x)\,\mathrm{d}x-\frac{f(a)-f(b)}{2}(b-a)+\frac{f'(b)-f'(a)}{8}(b-a)^2\\ &\qquad\le \frac{M(b-a)^3}{24}-\frac{M(b-a)^3}{8}Q^2-\frac{f'(b)-f'(a)}{8}(b-a)^2Q^2,\end{aligned}$$

where Q^2 is defined in (4).

Consequently, the inequality (3) holds.

Corollary 1. *Let the assumptions of Theorem* 3 *hold. If* f *satisfies* $f'(a) = f'(b) = 0$, *then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - \frac{f(a)+f(b)}{2} \right| \le \frac{M(b-a)^2}{24} - \frac{1}{2} \cdot \frac{\big(f(b)-f(a)\big)^2}{M(b-a)^2}.$$

Corollary 2. *Let the assumptions of Theorem* 3 *hold. If* f *satisfies* $f'(a) = f'(b)$, *then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - \frac{f(a)+f(b)}{2} \right| \le \frac{M(b-a)^2}{24} - \frac{1}{2M} \left(f'(a) - \frac{f(b)-f(a)}{b-a} \right)^2.$$

It should be noted that Corollary 1 and Corollary 2 provide improvements of the results given in [**8**].

Theorem 4. *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable function such that* f' *is integrable on* $[a,b]$ *and for all* $x \in [a,b]$ *with* $M > m$ *we have*

$$(10) \qquad m \le \frac{f'(x)-f'(x)}{x-a} \le M \quad \text{and} \quad m \le \frac{f'(b)-f'(x)}{b-x} \le M.$$

Then

$$(11) \qquad \Bigg| \int_a^b f(x)\,\mathrm{d}x - \frac{f(a)+f(b)}{2}(b-a) + \frac{1+P^2}{8}\big(f'(b)-f'(a)\big)(b-a)^2 - \frac{1+3P^2}{48}(m+M)(b-a)^3 \Bigg| \le \frac{(M-m)(b-a)^3}{48}(1-3P^2),$$

where

$$(12) \qquad P^2 = \frac{\left(f'(a) + f'(b) - 2\,\dfrac{f(b)-f(a)}{b-a} \right)^2}{\left(\dfrac{M-m}{2}\right)^2 (b-a)^2 - \left(f'(b) - f'(a) - \dfrac{m+M}{2}(b-a) \right)^2}.$$

Proof. It is clear that condition (10) can be given as

$$|k'(x) - k'(a)| \le M_1(x-a) \quad \text{and} \quad |k'(x) - k'(b)| \le M_1(b-x),$$

where $k'(x) = f'(x) - \dfrac{m+M}{2}x$ and $M_1 = \dfrac{M-m}{2}$. So we apply Theorem 3 on k, i.e., using the inequality (3) for $k(x) = f(x) - \dfrac{m+M}{4}x^2$ and M_1, we shall obtain the inequality (11) with (12).

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Institute of Applied Mathematics,
Faculty of Science,
Anshan University of Sciense and Technology,
Anshan 114044,
Liaoning, China

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