Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 16 (2005), 26–28.
Available electronically at **http: //pefmath.etf.bg.ac.yu**

COMPLETE MONOTONICITY PROPERTIES FOR A RATIO OF GAMMA FUNCTIONS

Chao-Ping Chen

Define for $x > 0$

$$F(x) = \frac{\Gamma(2x)}{x\Gamma^2(x)} \text{ and } G(x) = \frac{\Gamma(2x)}{\Gamma^2(x)}.$$

In this paper, we consider the logarithmically complete monotonicity properties for the functions F and $1/G$.

The gamma function is defined for $\operatorname{Re} z > 0$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t. \tag{1}$$

The psi or digamma function, the logarithmic derivative of the gamma function, can be expressed as

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-zt}}{1 - e^{-t}}\,\mathrm{d}t, \tag{2}$$

where $\gamma = 0.57721566490153286\ldots$ is the Euler-Mascheroni constant.

In 1997, Merkle [**1**] showed that the function $F(x) = \dfrac{\Gamma(2x)}{x\Gamma^2(x)}$ is strictly log-convex and the function $G(x) = \dfrac{\Gamma(2x)}{\Gamma^2(x)}$ is strictly log-concave on $(0, \infty)$. In this paper, we extend the results given by Merkle; we consider the logarithmically complete monotonicity properties for the functions F and $1/G$. Recall that a function f is said to be completely monotonic on an interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I; n = 0, 1, 2, \ldots). \tag{3}$$

If the inequality (3) is strict, then f is said to be strictly completely monotonic on I.

2000 Mathematics Subject Classification: 33B15, 26A48
Keywords and Phrases: Gamma function, psi function, logarithmically completely monotonic function.

Definition. *A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies*

$$(-1)^n \big(\ln f(x)\big)^{(n)} \geq 0 \quad (x \in I; n = 1, 2, \ldots). \tag{4}$$

If inequality (4) is strict for all $x \in I$ and for all $n \geq 1$, then f is said to be strictly logarithmically completely monotonic.

This definition was introduced in [**2**] by F. Qi and B.-N. Guo. Moreover, the authors showed that a (strictly) logarithmically completely monotonic function must be (strictly) completely monotonic.

The purpose of this paper is to establish the following result.

Theorem. *Let $I = (0, +\infty)$ and let $F(x) = \dfrac{\Gamma(2x)}{x\Gamma^2(x)}$, $G(x) = \dfrac{\Gamma(2x)}{\Gamma^2(x)}$, $x \in I$. Then we have*

(A) $(\ln F(x))' > 0$, $x \in I$,
(B) $(-1)^n \big(\ln F(x)\big)^{(n)} > 0$ *for $x \in I$ and $n = 2, 3, \ldots,$*
(C) *The function $1/G$ is strictly logarithmically completely monotonic on I.*

Proof. Using the duplication formula and the translation formula for the gamma function

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\,\Gamma(x)\Gamma(x + 1/2), \tag{5}$$

$$\Gamma(x + 1) = x\Gamma(x), \tag{6}$$

we conclude that

$$F(x) = \frac{2^{2x-1}\Gamma(x + 1/2)}{\sqrt{\pi}\,\Gamma(x + 1)}\,.$$

Taking logarithm and differentiation yields

$$\begin{aligned}\big(\ln F(x)\big)' &= 2\ln 2 + \psi(x + 1/2) - \psi(x + 1)\\ &= 2\ln 2 + \int_0^\infty \frac{e^{-(x+1)t} - e^{-(x+1/2)t}}{1 - e^{-t}}\,\mathrm{d}t\\ &= 2\ln 2 - \int_0^\infty \frac{e^{-xt}}{1 + e^{t/2}}\,\mathrm{d}t\end{aligned}$$

and therefore

$$(-1)^n \big(\ln F(x)\big)^{(n)} = \int_0^\infty \frac{t^{n-1}}{1 + e^{t/2}} e^{-xt}\,\mathrm{d}t > 0 \quad (x > 0;\ n = 2, 3, \ldots).$$

Clearly, $\big(\ln F(x)\big)'' > 0$, and then the function $x \mapsto \big(\ln F(x)\big)'$ is strictly increasing on $(0,\infty)$, which implies for $x>0$

$$\big(\ln F(x)\big)' > \big(\ln F(x)\big)'_{x=0} = 2\ln 2 - \int_0^\infty \frac{1}{1+e^{t/2}}\,\mathrm{d}t$$
$$= 2\ln 2 + 2\int_0^\infty \frac{1}{1+e^{-t/2}}\,\mathrm{d}\,(1+e^{-t/2}) = 0.$$

Using (5) and (6) we conclude that

$$G(x) = \frac{2^{2x-1}\Gamma(x+1/2)}{\sqrt{\pi}\,\Gamma(x)}.$$

Taking logarithm and differentiation yields

$$\begin{aligned}\Big(\ln\big(1/G(x)\big)\Big)' &= -2\ln 2 - \psi(x+1/2) + \psi(x)\\ &= -2\ln 2 - \int_0^\infty \frac{e^{-xt}-e^{-(x+1/2)t}}{1-e^{-t}}\,\mathrm{d}t\\ &= -2\ln 2 - \int_0^\infty \frac{e^{-xt}}{1+e^{-t/2}}\,\mathrm{d}t < 0\end{aligned}$$

and therefore

$$(-1)^n\Big(\ln\big(1/G(x)\big)\Big)^{(n)} = \int_0^\infty \frac{t^{n-1}}{1+e^{-t/2}}\,e^{-xt}\,\mathrm{d}t > 0 \quad (x>0;\ n=2,3,\ldots).$$

The proof is complete.

REFERENCES

1. M. Merkle: *On log-convexity of a ratio of gamma functions.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., **8** (1997), 114–119.
2. Feng Qi, Bai-Ni Guo: *Complete Monotonicities of Functions Involving the Gamma and Digamma Functions.* RGMIA Res. Rep. Coll., **7** (2004), no. 1, Art. 8. Available online at `http://rgmia.vu.edu.au/v7n1.html.`

Department of Applied Mathematics and Informatics,
Henan Polytechnic University,
Jiaozuo City, Henan, 454000,
People's Republic of China
E–mail: chenchaoping@hpu.edu.cn

(Received April 3, 2004)