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A RANDOM COEFFICIENT AUTOREGRESSIVE MODEL (RCAR(1)MODEL)

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A first order autoregressive model AR(1) with random coefficients is considered in this paper. Properties for such models are found. Some results of simulations Monte Carlo are reported.

1. INTRODUCTION

Models with random parameters are frequently used in time series analysis. Equation (1) of Section 2 constitutes our model which will be called RCAR(p), synonymous for *random coefficient autoregressive model of order p*. Our aim is to obtain some information on the distribution of RCAR(1) model.

2. DESCRIPTION OF MODEL

An autoregressive model of order p with random coefficients (by the name of RCAR(p) $*\,(m_1, m_2, \ldots, m_p; n)$) for time series X_t, $t \in D = \{\ldots, -1, 0, 1, \ldots\}$ is the model

$$X_t = A_1 X_{t-1} + A_2 X_{t-2} + \cdots + A_p X_{t-p} + B\xi_t\,, \qquad t \in D. \tag{1}$$

Coefficients $A_1, A_2, \ldots, A_p$ are discrete random variables which takes $m_1, m_2, \ldots, m_p$ different values and B is a random variable with n different values. We assume that the following assumptions are fulfilled:

$$\{\xi_t,\, t \in D\} \quad \text{is a white noise;} \tag{2}$$

$$A_1, A_2, \ldots, A_p, B \quad \text{are independent;} \tag{3}$$

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$$A_1, A_2, \ldots, A_p, B \quad \text{are independent of } \xi_t, \text{ all } t; \tag{4}$$

$$A_1, A_2, \ldots, A_p, B \quad \text{are independent of } X_t, \text{ all } t; \tag{5}$$

$$X_t \quad \text{is independent of } \xi_s, \text{ all } s > t; \tag{6}$$

For a stationary sequence $\{X_t,\, t \in D\}$, the RCAR(1)*(m;n) model is given by:

$$X_t = AX_{t-1} + B\xi_t\,, \quad t \in D. \tag{7}$$

We have $P(0 \le A < 1 - \varepsilon) = 1$ for $0 < \varepsilon < 1$, so we can write X_t in the form

$$X_t = \sum_{j=0}^{\infty} BA^j \xi_{t-j}\,, \quad t \in D. \tag{8}$$

Mean, variance and correlation function for series (8) can be established in the following way:

$$\begin{aligned} EX_t &= \sum_{j=0}^{\infty} E(B)E(A^j)E(\xi_{t-j}) = E(B)E(\xi)\sum_{j=0}^{\infty} E(A^j) \\ &= E(B)E(\xi)E\left(\frac{1}{1-A}\right), \end{aligned} \tag{9}$$

$$\begin{aligned} DX_t = E(X_t - EX_t)^2 &= E\Bigg(\sum_{j=0}^{\infty}\big(BA^j\xi_{t-j} - E(B)E(A^j)E(\xi_{t-j})\big)\Bigg)^2 \\ &= \sum_{j=0}^{\infty} E\Big(BA^j\xi_{t-j} - E(B)E(A^j)E(\xi_{t-j})\Big)^2 \\ &\quad + \sum_{j,i=0, j\ne i}^{\infty} E\big(BA^j\xi_{t-j} - E(B)E(A^j)E(\xi_{t-j})\big) \\ &\qquad \cdot\Big(BA^i\xi_{t-i} - E(B)E(A^i)E(\xi_{t-i})\Big) \\ &= \Big(E(B^2)D\xi + \big(D(B)\big)(E\xi)^2\Big)E\left(\frac{1}{1-A^2}\right) \\ &\quad + \big(E(B)\big)^2(E\xi)^2 D\left(\frac{1}{1-A}\right), \end{aligned} \tag{10}$$

$$\begin{aligned} K(t, t+\tau) &= E\sum_{j=0}^{\infty}\Big(BA^j\xi_{t-j} - E(B)E(A^j)E(\xi_{t-j})\Big) \\ &\quad \cdot\sum_{v=0}^{\infty}\Big(BA^v\cdot\xi_{t+\tau-v} - E(B)E(A^v)E(\xi_{t+\tau-v})\Big) \\ &= \big(E(B^2)D\xi + D(B)(E\xi)^2\big)E\left(\frac{A^\tau}{1-A^2}\right) \\ &\quad + \big(E(B)\big)^2(E\xi)^2\cdot D\left(\frac{1}{1-A}\right), \qquad \tau = 0, \pm 1, \pm 2, \ldots \end{aligned} \tag{11}$$

3. RCAR(1)*(2;2) MODEL

In this section we shall deal with the following model for a stationary sequence $\{X_t, t \in D\}$:

$$X_t = AX_{t-1} + B\xi_t\,, \quad t \in D, \tag{12}$$

A and B are random coefficients with distributions

$$A : \begin{pmatrix} \alpha & \beta \\ p_1 & q_1 \end{pmatrix}, 0 < \alpha < \beta < 1, \qquad B : \begin{pmatrix} 1 & 0 \\ p_2 & q_2 \end{pmatrix},$$

where p_1, q_1, p_2, q_2 are probabilities with $p_1 + q_1 = 1$ and $p_2 + q_2 = 1$. From (9), (10), (11) follows:

$$EX_t = (E\xi) \cdot p_2 \cdot \left(\frac{p_1}{1-\alpha} + \frac{q_1}{1-\beta} \right), \tag{13}$$

$$\begin{aligned} DX_t = {} & \left(p_2 D\xi + p_2 q_2 (E\xi)^2\right) \left(\frac{p_1}{1-\alpha^2} + \frac{q_1}{1-\beta^2} \right) \\ & + p_2^2 (E\xi)^2 p_1 q_1 \left(\frac{1}{1-\alpha} - \frac{1}{1-\beta} \right)^2, \end{aligned} \tag{14}$$

$$\begin{aligned} \gamma_\tau = K(t, t+\tau) = {} & \left(p_2 D\xi + p_2 q_2 (E\xi)^2\right) \left(\frac{p_1 \alpha^\tau}{1-\alpha^2} + \frac{q_1 \beta^\tau}{1-\beta^2} \right) \\ & + p_2^2 (E\xi)^2 p_1 q_1 \left(\frac{1}{1-\alpha} - \frac{1}{1-\beta} \right)^2 \qquad \tau = \pm 1, \pm 2, \ldots \\ \gamma_o \equiv {} & DX_t. \end{aligned}$$

For model (12) which can be written in the form (it holds $p_1 = p_2$):

$$X_{t+1} = \begin{cases} \alpha X_t, & \text{w.p.} \quad p_1 q_1 \\ \beta X_t, & \text{w.p.} \quad q_1^2 \\ \alpha X_t + \xi_{t+1}, & \text{w.p.} \quad p_1^2 \\ \beta X_t + \xi_{t+1}, & \text{w.p.} \quad p_1 q_1 \end{cases} \tag{15}$$

we can consider the conditional probability:

$$\psi(s|s_t) = P(X_{t+1} < s | s_t \le X_t < s_t + h) \tag{16}$$

and the conditional density is obtained as $h \to 0$ by:

$$\begin{aligned} g(s|s_t) & = \frac{\mathrm{d}}{\mathrm{d}s} P(X_{t+1} < s | X_t = s_t) \\ & = p_1 q_1 \delta(s - \alpha s_t) + {q_1}^2 \delta(s - \beta s_t) + {p_1}^2 g_\xi(s - \alpha s_t) H(s - \alpha s_t) \cdot \\ & + p_1 q_1 g_\xi(s - \beta s_t) H(s - \beta s_t) \end{aligned} \tag{17}$$

Here $\delta(\cdot)$ is the DIRAC delta function, $H(\cdot)$ is HEAVISIDE function defined by:

$$H_{(0)}(s-\alpha s_t) = \begin{cases} 1, & s=\alpha s_t \\ 0, & s\neq \alpha s_t \end{cases},$$

$$H(s-\alpha s_t) = H_{(0,\infty)}(s-\alpha s_t) = \begin{cases} 1, & s>\alpha s_t \\ 0, & s\leq \alpha s_t \end{cases},$$

and $g_\xi(\cdot)$ is the density of random variables ξ_t, for all t.

We can write (17) in the form:

$$\begin{aligned} g(s|s_t) = \big(p_1 q_1 \delta(s-\alpha s_t)\big)^{H_{(0)}(s-\alpha s_t)} \cdot \big(q_1^2 \delta(s-\beta s_t)\big)^{H_{(0)}(s-\beta s_t)} \\ \cdot\big(p_1^2 g_\xi(s-\alpha s_t)^{H(s-\alpha s_t)} \cdot \big(p_1 q_1 g_\xi(s-\beta s_t)\big)^{H(s-\beta s_t)}. \end{aligned} \tag{18}$$

Having observed $(X_2,\dots,X_{n+1})$ and fixed $X_1=s_1$ from the model (15) we can estimate the parameters p_1, q_1 of model (15) using conditional likelihood function:

$$L(p_1) = \prod_{t=1}^{n} g(s_{t+1}|s_t). \tag{19}$$

For fixed α,β the maximum likelihood estimators of p_1 and q_1 are:

$$\hat{p}_1 = \frac{A_1+2C_1+D_1}{2(A_1+B_1+C_1+D_1)}, \tag{20}$$

$$\hat{q}_1 = 1-\hat{p}_1 = \frac{A_1+2B_1+D_1}{2(A_1+B_1+C_1+D_1)}, \tag{21}$$

where

$$A_1 = \sum_{t=1}^{n} H_{(0)}(s_{t+1}-\alpha s_t), \qquad B_1 = \sum_{t=1}^{n} H_{(o)}(s_{t+1}-\beta s_t),$$

$$C_1 = \sum_{t=1}^{n} H_{(0,\infty)}(s_{t+1}-\alpha s_t), \qquad D_1 = \sum_{t=1}^{n} H_{(0,\infty)}(s_{t+1}-\beta s_t).$$

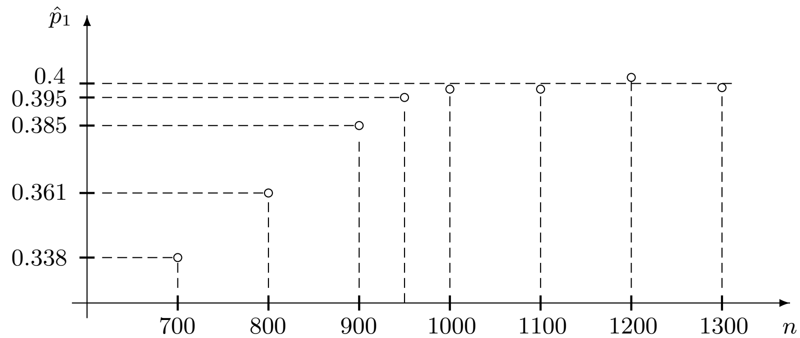
4. SIMULATION MONTE CARLO

Let us choose arbitrary values $\alpha=0,2$, $\beta=0,4$, $p_1=0,4$ and assume that exponential distribution with mean 2 is used for sequence $\{\xi_t\}$. We can find n random variables $\gamma_k(k=2,\dots,n+1)$ over (0,1) and with transformation

$$\xi_k = -2\cdot\ln(\gamma_k), \qquad k=2,\dots,n+1$$

we get n random values with exponential distribution. From (15) we can get n modelled values of random variable X_t.

If we use that values we can find $\hat{p}_1$ from (20) for different values of n.



Values of the estimator $\hat{p}_1$

The results of simulation for various values of n provide the estimates very close to the value of p_1.

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