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APPELL POLYNOMIALS AND LOGARITHMIC CONVEXITY

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In this article we give necessary and sufficient conditions for logarithmic convexity of some sequences of Appell polynomials. Then we apply our results to Turan's type polynomial inequalities. Precise upper and lower bounds for this class of polynomials are also determined and asymptotic behavior of $A_n(x)^{1/n}$ $(n \to \infty)$ as well.

1. INTRODUCTION

A real sequence $\{a_n\}$, $n = 0, 1, 2, \ldots$ generates a sequence $\{A_n(x)\}$ of Appell polynomials defined by:

$$A_n(x) := \sum_{k=0}^{n} a_k \binom{n}{k} x^{n-k} \qquad (n = 0, 1, 2, \ldots).$$

This class of polynomials is of importance in real and combinatorial analysis [**1**], [**2**]. For example, the classical Bernoulli and Laguerre polynomials belong to this class.

For an arbitrary generating sequence $\{a_n\}$ it is difficult to say anything about behavior of the sequence $\{A_n(x)\}$. But, as we shall see, logarithmic convexity is the property which closely connects $\{a_n\}$ and $\{A_n(x)\}$.

Hence, our aim here is to investigate relationships between logarithmic convexity (concavity) of the sequences $\{A_n(x)\}$ and $\{a_n\}$. For this purpose, we shall suppose throughout that the generating sequence $\{a_n\}$ consists of positive numbers.

2. RESULTS

We shall prove the following propositions:

Proposition 1. *The sequence* $\{\log A_n(x)\}$ *is convex* (*concave*) *for* $x > 0$ *if and only if the sequence* $\{\log a_n\}$ *is convex* (*concave*).

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Proposition 2. *The sequence* $\{B_n(x)\}$ *defined by*

$$B_n(x) := (A_n(x)/a_n)^{1/n}, \quad B_0(x) := 1 \qquad (n = 1, 2, \ldots)$$

is monotone non-decreasing (*non-increasing*) *for* $x > 0$ *if and only if the sequence* $\{\log a_n\}$ *is concave* (*convex*).

It is said that the sequence of polynomials $\{C_n(x)\}$ have T^+ property if it satisfies TURAN's inequality

$$C_n^{\,2}(x) - C_{n-1}(x)C_{n+1}(x) \geq 0 \qquad (x \in [a, b],\ n \in \mathbb{N}).$$

Analogously, $\{C_n(x)\}$ have T^- property if it satisfies inverse TURAN's inequality

$$C_n^{\,2}(x) - C_{n-1}(x)C_{n+1}(x) \leq 0 \qquad (x \in [a, b],\ n \in \mathbb{N}).$$

By the referee's opinion, an application of our results to TURAN's inequality is of importance. We shall give here some T property criteria. Note that again $\{a_n\}$ is a sequence of positive numbers.

Proposition 3. *The sequence* $\{A_n(x)\}$ *have* T^+ (T^-) *property for* $x \in (0, b], b > 0$ *if and only if the sequence* $\{a_n\}$ *have* T^+ (T^-) *property.*

Proposition 4. *If the sequence* $\{a_n\}$ *have not* T^+ (T^-) *property, then also* $\{A_n(x)\}$ *have not this property for* $x \in [a, b], a < 0 < b$.

Proposition 5. *If* $\{A_n(x)\},\ A_0(x) = a_0 = 1,$ *have* T^+ *property for* $x > 0$, *then*

$$\frac{a_n}{a_1^n} \leq \frac{A_n(x)}{A_1^n(x)} \leq 1 \qquad (x > 0,\ n \in \mathbb{N}).$$

If $\{A_n(x)\}$ *have* T^- *property then the reverse inequalities hold.*

Proposition 6. *Define* $A_n^{(\theta)}(x) := \sum_k a_k^{\theta}\binom{n}{k}x^{n-k},\ \theta \in \mathbb{R},\ n \in \mathbb{N};\ A_n^{(1)}(x) = A_n(x)$.

If $\{A_n(x)\}$ *have* T^+ *property for* $x > 0$, *then* $\{A_n^{(\theta)}(x)\}$ *have* T^+ *property for* $\theta \geq 0$ *and* T^- *property for* $\theta < 0$.

Analogoues statement takes place if $\{A_n(x)\}$ *have* T^- *property.*

Proposition 7. (i) *If* $\{A_n(x)/a_n\}$ *have* T^+ *property for* $x > 0$, *then* $\{A_n(x)\}$ *have* T^- *property for* $x > 0$.

(ii) *If* $\{A_n(x)/a_n\}$ *have* T^- *property for* $x > 0$, *then* $\{A_n(x)\}$ *have* T^+ *property for* $x > 0$.

Supposing logarithmic convexity (concavity) on the sequence $\{a_n\}$, enables us to get some control over $\{A_n(x)\},\ x > 0$. This is shown in the sequel.

Apart from the assertion in Proposition 5, we get much stronger inequalities for $A_n(x)$ in the next

Proposition 8. *If the sequence* $\{\log a_n\}$ *is convex, then*

$$a_n\big(1+(a_{n-1}/a_n)x\big)^n \le A_n(x) \le a_0(x+a_n/a_{n-1})^n \quad (x>0;\ n\in\mathbb{N}),$$

with equality if and only if $A_n(x)$ *is of the form* $A_n(x)=a_0(x+a)^n$, *for some positive* a.

If $\{\log a_n\}$ *is concave, then the reverse inequalities hold.*

Logarithmic convexity (concavity) of the sequence $\{a_n\}$ is equivalent to monotonicity of the sequence $\{a_n/a_{n+1}\}$. Hence $\lim_{n\to\infty} a_n/a_{n+1}$ exists (in wider sense), and we obtain asymptotic behavior of $\big(A_n(x)\big)^{1/n}$ $(n\to\infty)$.

Proposition 9. (i) *If the sequence* $\{a_n/a_{n+1}\}$ *is monotone and*

$$\lim_{n\to\infty} a_n/a_{n+1}=c \qquad (0<c<\infty),$$

then, for fixed $x>0$,

$$\big(A_n(x)\big)^{1/n} \to x+1/c \qquad (n\to\infty).$$

If $\{a_n\}$ *is of the form* $a_n=(\ell_n)^n$, *where* $\{\ell_n\}$ *is from the class of normalized slowly varying sequences in Karamata's sense (see definition below), we have*

(ii) *if* $a_n/a_{n+1}\uparrow^{\infty}$, *then* $\big(A_n(x)\big)^{1/n}\to x \qquad (n\to\infty)$;

(iii) *if* $a_n/a_{n+1}\downarrow 0$, *then* $\big(A_n(x)\big)^{1/n}\sim a_{n+1}/a_n\sim \ell_n \qquad (n\to\infty)$.

PROOFS

Proof of Proposition 1. Suppose first that the sequence $\big\{\log A_n(x)\big\}$ is convex (concave) for $x>0$. Then the polynomial $P(x)$:

$$\begin{aligned}P(x) &:= A_n^2(x)-A_{n-1}(x)A_{n+1}(x)\\ &= a_n^2-a_{n-1}a_{n+1}+(n-1)(a_na_{n-1}-a_{n+1}a_{n-2})x+\cdots+(a_1^2-a_0a_2)x^{2n-2}\end{aligned}$$

is non-negative (non-positive) for $x>0$. Using the identity

$$A'_n(x)=nA_{n-1}(x) \tag{1}$$

we obtain

$$P'(x)=(n-1)\big(A_n(x)A_{n-1}(x)-A_{n+1}(x)A_{n-2}(x)\big).$$

Hence, polynomials $P(x)$ and $P'(x)$ are of the same sign, i.e. $P(x)$ is either non-negative and non-decreasing or non-positive and non-increasing for $x>0$.

Since it is also continuous in x, it follows that $P(0)=a_n^2-a_{n-1}a_{n+1}$ has the same sign as $P(x),\ x>0$.

Suppose now that the sequence $\{\log a_n\}$ is convex (concave).

Putting $c_n = c_n(x) := a_n x^{-n},\ x > 0;\ n = 0, 1, 2, \ldots$, we have to prove that if $\{\log c_n\}$ is convex (concave) then the sequence $\{\log C_n\}$, where

$$C_n := \sum_{k=0}^{n} \binom{n}{k} c_k \qquad (n = 0, 1, 2, \ldots),$$

is also convex (concave).

It is not difficult to check that if $\{\log c_n\}$ is convex (concave), then the sequence $\{\log c_n^{(1)}\}$, defined by $c_n^{(1)} := c_n + c_{n-1}$, is also convex (concave).

By induction, the same is valid for sequences $\{\log c_n^{(m)}\}$, where

$$c_n^{(m+1)} := c_n^{(m)} + c_{n-1}^{(m)} \qquad (m = 1, 2, \ldots).$$

It is only left to note that $c_n^{(n)} = C_n$.

Remark 1. It is evident from the first part of the proof that convexity (concavity) of the sequence $\{\log A_n(x)\},\ x \in (0, b]$, where b is some positive constant, implies convexity (concavity) of $\{\log a_n\}$. From the other hand, convexity (concavity) of the sequence $\{\log a_n\}$ implies convexity (concavity) of $\{\log A_n(x)\}$ for all positive x.

Remark 2. As the referee notes, the second part of proposition is also proved in [**3**].

Proof of Proposition 2. Assume that the sequence $\{\log a_n\}$ is concave.

By Proposition 1 the sequence $\big\{\log A_n(t)\big\},\ t > 0$, is also concave i.e.

$$\frac{A_n(t)}{A_{n+1}(t)} \geq \frac{A_{n-1}(t)}{A_n(t)}$$

or, by (1),

$$\frac{1}{n+1}\,\frac{A'_{n+1}(t)}{A_{n+1}(t)} \geq \frac{1}{n}\,\frac{A'_n(t)}{A_n(t)}\,. \tag{2}$$

Integrating (2) over $t \in (0, x)$, we get

$$\big(A_{n+1}(x)/a_{n+1}\big)^{1/(n+1)} \geq \big(A_n(x)/a_n\big)^{1/n} \qquad (n = 1, 2, \ldots). \tag{3}$$

This means that the sequence $\big\{B_n(x)\big\}$ is monotone non-decreasing for each fixed $x > 0$.

The convex case can be treated similarly.

Suppose now that $\big\{B_n(x)\big\}$ is monotone and consider the polynomial $Q(x)$ defined by

$$Q(x) := \big(A_{n+1}(x)/a_{n+1}\big)^n - \big(A_n(x)/a_n\big)^{n+1}.$$

By assumption, $Q(x)$ is non-negative (non-positive) for all $x > 0$.

We have $Q(0) = 0$ and, by (1),

$$Q'(0) = n(n+1)(a_n/a_{n+1} - a_{n-1}/a_n) \qquad (n = 1, 2, \ldots).$$

Therefore,

$$Q(x) = n(n+1)\Big(\frac{a_n}{a_{n+1}} - \frac{a_{n-1}}{a_n}\Big)x + \cdots + \bigg(\Big(\frac{a_0}{a_{n+1}}\Big)^n - \Big(\frac{a_0}{a_n}\Big)^{n+1}\bigg)x^{n(n+1)}. \tag{4}$$

Since x is independent of n, we see from (4) that, for sufficiently small x, the signs of $Q'(0)$ and $Q(x),\ x > 0$ have to be the same, i.e. Proposition 2 is proved.

Proof of Proposition 3. It follows from Proposition 1 and Remark 1.

Proof of Proposition 4. This is a logical consequence of the previous proposition.

Proof of Proposition 5 needs the following lemma.

Lemma 1. *If the sequence $\{b_n\},\ b_0 = 1$ of positive numbers have T^+ property, then the sequence $\{b_n^{1/n}\}$ is non-increasing. Analogously, if $\{b_n\}$ have T^- property then $\{b_n^{1/n}\}$ is monotone non-decreasing.*

Proof. T^+ property implies $b_n^2 \geq b_{n-1}b_{n+1},\ n \in \mathbb{N}$. Hence

$$(b_0b_2)(b_1b_3)^2(b_2b_4)^3 \cdots (b_{n-1}b_{n+1})^n \leq b_1^2 b_2^4 b_3^6 \cdots b_n^{2n},$$

gives $b_{n+1}^n \leq b_n^{n+1}$, i.e. $\{b_n^{1/n}\}$ is non-increasing.

Proof of T^- case goes along the same lines.

Proof of Proposition 5. T^+ property and Lemma 1 imply $\{A_n(x)^{1/n}\}$ non-increasing. Therefore $A_n(x)^{1/n} \leq A_1(x)^1$ i.e. $A_n(x)/A_x^n(x) \leq 1$.

Otherwise, by Proposition 3 we obtain logarithmic concavity of the sequence $\{a_n\}$. Hence, from Proposition 2 we derive

$$(A_n(x)/a_n)^{1/n} \geq A_1(x)/a_1 \qquad (x > 0,\ n = 1, 2, \ldots).$$

This is exactly the left-hand side of the inequality from Proposition 5. The other case is treated similarly.

Proof of Proposition 6. This is a consequence of the fact that if $\{a_n\}$ have T^+ property, then $\{a_n^\theta\}$ have T^+ property for $\theta \geq 0$ and T^- property if $\theta < 0$. Now we can apply Proposition 3.

Proof of Proposition 7. We shall prove just the assertion (i). The case (ii) can be treated analogously.

Since $\{A_n(x)/a_n\}$ have T^+ property, Lema 1 asserts $\{(A_n(x)/a_n)^{1/n}\}$ non-increasing. By Proposition 2, this implies convexity of the sequence $\{\log a_n\}$ which in turn, by Proposition 1, gives convexity of $\{\log A_n(x)\}$.

This is equivalent with T^- property for the sequence $\{A_n(x)\}$.

Proof of Proposition 8. Suppose that $\{\log a_n\}$ is concave. To obtain the right-hand side, we proceed from (3) and, replacing $A_n(t)$ with $A'_{n+1}(t)/(n+1)$, we get

$$\big(A_{n+1}(t)\big)^{-n/(n+1)}A'_{n+1}(t) \le (n+1)a_n(a_{n+1})^{-n/(n+1)}.$$

Integrating over $t \in (0,x)$, we obtain the needed inequality.

Since logarithmic concavity of $\{a_n\}$ implies non-decreasing of the sequence $\{a_{n-1}/a_n\}$, we get

$$\frac{a_{n-1}}{a_n} \ge \frac{a_{k-1}}{a_k} \qquad (1 \le k \le n).$$

From there follows

$$a_k \ge a_0\Big(\frac{a_n}{a_{n-1}}\Big)^k \qquad (0 \le k \le n).$$

Hence

$$A_n(x) := \sum_k a_k\binom{n}{k}x^{n-k} \ge a_0\sum_k\Big(\frac{a_n}{a_{n-1}}\Big)^k\binom{n}{k}x^{n-k},$$

i.e. $A_n(x) \ge a_0(x + a_n/a_{n-1})^n$.

This is exactly the left-hand side of the target inequality.

It is evident that the reversed inequalities take place if we suppose logarithmic convexity of the sequence (a_n).

Note also that the equality sign in the statement of Proposition 8 holds if and only if it holds in (3) i.e. if $A_n(x)$ is of the mentioned form.

Proof of Proposition 9. If, for example,

$$a_{n-1}/a_n \downarrow c \qquad \big(0 < c < \infty \quad (n \to \infty)\big),$$

rewrite the inequality from previous proposition in the form

$$\frac{\sqrt[n]{a_n}}{(a_n/a_{n-1})}\,(x + a_n/a_{n-1}) \le A_n(x)^{1/n} \le \sqrt[n]{a_0}\,(x + a_n/a_{n-1}). \tag{5}$$

Then $a_n/a_{n-1} \to 1/c$ and also (as is well known) $\sqrt[n]{a_n} \to 1/c$.

Hence, from (5), we obtain the proof of part (i).

Extreme cases $c = 0, c = \infty$ can be treated in the following way.

It is obvious from (5) that necessary condition for asymptotic equivalence is

$$\sqrt[n]{a_n} \sim \frac{a_n}{a_{n-1}} \qquad (n \to \infty). \tag{6}$$

The form of sequences satisfying (6) is given by the following lemma.

Lemma 2. *A sequence $\{a_n\}$ of positive numbers satisfies the condition* (6) *if and only if $a_n = (\ell_n)^n$, where $\{\ell_n\}$ is from the class of normalized slowly varying sequences* (NSVS) *in Karamata's sense.*

Definition. *A sequence* $\{\ell_n\}$ *is* NSVS *if and only if it can be represented in the form*

$$\ell_n = C \exp\Big(\sum_{i=1}^{n} \epsilon_i/i\Big),$$

where $C > 0$ *and* $\{\epsilon_n\}$ *is a null sequence* [**4**, p. 53].

Properties. If $\{\ell_n\}$ is NSVS, then

(i) $\ell_{n+1} \sim \ell_n$; (ii) $n\Big(\dfrac{\ell_{n+1}}{\ell_n} - 1\Big) \to 0$; (iii) $\forall\lambda > 0,\ \ell_{[\lambda n]} \sim \ell_n \quad (n \to \infty)$;

(iv) the sum, product and quotient of two NSVS is also NSVS.

Examples of slowly varying sequences $\{\ell_n\}$ satisfying conditions of the part (iii) of Proposition 9 are

$$\{\log^a(n+2)\},\ \ a > 0 \text{ or } \big\{\exp\big(\log^b(n+1)\big)\big\},\ 0 < b < 1.$$

For the part (ii) we can take $1/\ell_n$ instead of ℓ_n.

Proof of Lemma 2. Putting $a_n = \exp(nb_n)$, the relation (6) becomes

$$n(b_n - b_{n-1}) \to 0 \qquad (n \to \infty).$$

Define $\epsilon_n := n(b_n - b_{n-1}),\ \ n \in \mathbb{N}$. Then $\{\epsilon_n\}$ is a null sequence and

$$b_n = b_0 + \sum_{i=1}^{n} \epsilon_i/i.$$

Therefore, by the above definition,

$$\exp b_n = e^{b_0} \exp\Big(\sum_{i=1}^{n} \epsilon_i/i\Big) = \ell_n,$$

is NSVS, i.e. $a_n = (\exp b_n)^n = (\ell_n)^n$.

On the other hand, if $a_n = (\ell_n)^n$, using the above Properties we obtain

$$\frac{a_n}{a_{n-1}} = \ell_n(\ell_n/\ell_{n-1})^{n-1} = \ell_n \exp((n-1)\epsilon_n/n) \sim \ell_n = \sqrt[n]{a_n} \qquad (n \to \infty).$$

Therefore Lemma 2 is proved and the proof of the part (ii) follows from (5) at once.

For the proof of the part (iii), note that now $\{a_n\}$ is logarithmicaly convex and appropriate inequality from Proposition 8 is

$$\frac{\sqrt[n]{a_n}}{(a_n/a_{n-1})}\big(1 + (a_{n-1}/a_n)\,x\big) \le \frac{A_n(x)^{1/n}}{(a_n/a_{n-1})} \le \sqrt[n]{a_0}\,\big(1 + (a_{n-1}/a_n)\,x\big).$$

From there, taking in account Lemma 2, proof of (iii) follows.

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