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APPLICATIONS OF THE HYPER-POWER METHOD FOR COMPUTING MATRIX PRODUCTS

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We introduce representations for $\{1,2,3\}$, $\{1,2,4\}$-inverses in terms of matrix products involving the MOORE-PENROSE inverse. We also use representations of $\{2,3\}$ and $\{2,4\}$-inverses of a prescribed rank, introduced in [**6**] and [**9**]. These representations can be computed by means of the modification of the hyper-power iterative process which is used in computing matrix products involving the MOORE-PENROSE inverse, introduced in [**8**]. Introduced methods have arbitrary high orders $q \geq 2$. A few comparisons with the known modification of the hyper-power method from [**17**] are presented.

1. INTRODUCTION

Let $\mathbb{C}^n$ be the n-dimensional complex vector space, $\mathbb{C}^{m\times n}$ the set of $m \times n$ complex matrices, and $\mathbb{C}_r^{m\times n} = \{X \in \mathbb{C}^{m\times n} \ : \ \operatorname{rank}(X) = r\}$. We use $\mathcal{N}(A)$ to denote the kernel and $\mathcal{R}(A)$ to denote the range of A, and $\rho(A)$ to denote the spectral radius of A. If $A \in \mathbb{C}^{n\times n}$ and L, M are complementary subspaces of $\mathbb{C}^n$, then $P_{L,M}$ denotes the projector on L along M.

For any $A \in \mathbb{C}^{m\times n}$ PENROSE defined the following equations in X:

$$(1)\ AXA=A, \quad (2)\ XAX=X, \quad (3)\ (AX)^*=AX, \quad (4)\ (XA)^*=XA.$$

For a subset $\mathcal{S}$ of the set $\{1,2,3,4\}$ the set of matrices obeying the conditions represented in $\mathcal{S}$ will be denoted by $A\{\mathcal{S}\}$. A matrix G in $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of A and denoted by $A^{(\mathcal{S})}$. In particular, the set $A\{1,2,3,4\}$ consists of a single element, the MOORE-PENROSE inverse of A, denoted by $A^{\dagger}$. The set of $\{2,3\}$ and $\{2,4\}$-inverses of a given rank $0 < s < r$ is denoted by $A\{2,3\}_s$ and $A\{2,4\}_s$, as in [**5**], [**6**] and [**9**].

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An application of the following *hyper-power method* of the order 2

$$X_{k+1} = X_k(2I - AX_k) = (2I - X_k A)X_k$$

in usual matrix inversion dates back to the well-known paper of SCHULZ [**15**]. BEN-ISRAEL and COHEN shown that this iterative process converges to $A^\dagger$ provided that $X_0 = \alpha A^*$, where α is a positive and sufficiently small real number [**2**], [**3**], [**4**]. The hyper-power iterative method of an arbitrary order $q \geq 2$ was originally devised by ALTMAN [**1**] for inverting a nonsingular bounded operator in a BANACH space. In [**11**] the convergence of the same method is proved under the condition which is weaker than the one assumed in [**1**], and better error estimates are derived.

ZLOBEC in [**21**] defined two hyper-power iterative methods of an arbitrary high order $q \geq 2$:

$$\begin{aligned} Y_0 &= \alpha A^*, \\ T_k &= I - Y_k A, \\ M_k &= I + T_k + \cdots + T_k^{q-1} \\ Y_{k+1} &= M_k Y_k, \quad k = 0, 1, \ldots \end{aligned} \tag{1.1}$$

$$\begin{aligned} Y_0' &= \alpha A^*, \\ {T_k}' &= I - A{Y_k}', \\ M_k' &= I + T_k' + \cdots + {T_k'}^{q-1} \\ {Y_{k+1}}' &= {Y_k}' M_k', \quad k = 0, 1, \ldots \end{aligned} \tag{1.1$'$}$$

It is well known that if we take

$$0 < \alpha \leq \frac{2}{\operatorname{Tr} A^* A},$$

then $Y_k \to A^\dagger$ and $Y_k' \to A^\dagger$ [**21**].

If A is $m \times N$ complex matrix, then the process (1.1) is superior with respect to (1.1') when $m > N$ [**8**].

The hyper-power iterative method of the order 2 is investigated in [**16**] in view of the singular value decomposition of A. Recently, this method is investigated in [**11**] and [**13**]. In [**13**] several error estimates of the method are investigated. In [**11**] the hyper-power method of the order 2 is implemented by means of parallel computing, and several acceleration procedures are introduced.

In [**20**] there are given necessary and sufficient conditions for the starting approximation of the hyper-power iterative method, ensuring the convergence of these methods to an arbitrary $\{1,2\}$-inverse. Modifications of the hyper-power method for computing various subclasses of $\{1,2\}$-inverses are introduced in [**17**].

In [**8**] are introduced two methods for computing the matrix products $A^\dagger B$ and $BA^\dagger$, involving the MOORE-PENROSE inverse, where $A \in \mathbb{C}^{m\times N}$ and $B \in$

$\mathbb{C}^{m\times n}$ are arbitrary complex matrices with equal number of rows. The starting matrix Y_0 is chosen such that

$$(1.2)\qquad \begin{array}{l} Y_0 = A^*WA^*, \text{ for some } W \in \mathbb{C}^{m\times N} \text{ provided that} \\ \rho\left(P_{\mathcal{R}(A)} - AY_0\right) < 1, \end{array}$$

where $P_{\mathcal{R}(A)}$ is the orthogonal projection on the range of A. The sequence $\{X_k\}$, defined by the following modification of the hyper-power method:

$$(1.3)\qquad \begin{aligned} Y_0 &\text{ is given by (1.2)},\\ X_0 &= Y_0B,\\ T_0 &= I - Y_0A,\\ M_k &= I + T_k + T_k^2 + \ldots + T_k^{q-1},\\ X_{k+1} &= M_kX_k,\\ T_{k+1} &= T_k^q = I + M_k[T_k - I]. \end{aligned}$$

converges to $A^\dagger B$ [**8**].

In [**8**] it is shown that (1.3) is an improvement (over using (1.1) to find $A^\dagger$ and then forming $A^\dagger B$) only when $N > n$.

In [**19**] we develop an iterative method for computing the *best approximate solution* and the *basic solution* of a given system of linear equations. This method is an adaptation of the modified hyper-power method (1.3). In this paper we introduce several modifications of the iterative process (1.3), applicable in computing $\{1,2,3\}$, $\{1,2,4\}$ and $\{2,3\}$, $\{2,4\}$ generalized inverses of a given rank.

In the second section we introduce representations for $\{1,2,3\}$ and $\{1,2,4\}$ inverses of a given complex matrix, in terms of matrix products involving the Moore-Penrose inverse. We also restate usual representations for $\{2,3\}$ and $\{2,4\}$-inverses from [**5**], [**6**] and [**9**].

In view of these representations, we propose several modifications of the hyper-power method (1.3), which can be used in computation of $\{2,3\}$, $\{1,2,3\}$ and $\{2,4\}$, $\{1,2,4\}$-inverses. Methods have arbitrary high order $q \geq 2$. Representations for $\{i,j,k\}$ inverses of a matrix of rank 1 are also investigated. Introduced methods can be considered as a continuation of the papers [**8**] and [**19**].

In the third section we describe main implementation details in the package MATHEMATICA and present an illustrative example.

2. ITERATIVE METHODS FOR COMPUTING $\{i,j,k\}$ INVERSES

The following representations for $\{2,3\}$ $\{2,4\}$-inverses are restated from [**5**, p. 56–58], [**6**, p. 47–48] and [**9**].

Proposition 2.1. *Let $A \in \mathbb{C}_r^{m\times N}$ and $0 < s < r$ be a chosen integer. Then the following is valid*:

(a) $A\{2,4\}_s = \{(W_2A)^\dagger W_2 : \quad W_2 \in \mathbb{C}^{s\times m},\ W_2A \in \mathbb{C}_s^{s\times N}\}$.

(b) $A\{2,3\}_s = \{W_1(AW_1)^\dagger : \quad W_1 \in \mathbb{C}^{N\times s},\ AW_1 \in \mathbb{C}_s^{m\times s}\}$.

In the following theorem we investigate similar representations of $\{1,2,3\}$ and $\{1,2,4\}$-inverses, in terms of matrix products involving the Moore-Penrose inverse.

Theorem 2.1. *Let $A \in \mathbb{C}_r^{m\times N}$ and $A = PQ$ be a full-rank factorization of A. Then the following statements about the sets $A\{1,2,3\}$ and $A\{1,2,3\}$ are valid*:

(a) *The set of $\{1,2,4\}$-inverses of A can be represented as follows*:

$$A\{1,2,4\} = \left\{(W_2A)^\dagger W_2 : \quad W_2 \in \mathbb{C}^{r\times m}, \quad W_2P \text{ is invertible}\right\}.$$

(b) *The set of $\{1,2,3\}$-inverses of A can be represented as follows:*

$$A\{1,2,3\} = \left\{W_1(AW_1)^\dagger : \quad W_1 \in \mathbb{C}^{N\times r}, \quad QW_1 \text{ is invertible}\right\}.$$

(c) *Particularly*,

$$A^\dagger = (P^*A)^\dagger P^* = Q^*(AQ^*)^\dagger.$$

Proof. **(a)** Consider an arbitrary matrix $W_2 \in \mathbb{C}^{r\times m}$, such that W_2P is invertible. Since the matrix $X = (W_2A)^\dagger W_2$ is $\{2,4\}$ inverse of A, we must to verify the equation $AXA = A$. We use the following important property of the Moore–Penrose inverse [**7**]: $(UV)^\dagger = V^\dagger U^\dagger$ if and only if both of the following two conditions are satisfied

$$U^\dagger UVV^*U^* = VV^*U^*, \quad VV^\dagger U^*UV = U^*UV. \tag{2.1}$$

The matrix $U = W_2P$ is invertible and $V = Q$ is the right invertible. So the conditions (2.1) are satisfied in this case, and we get

$$AXA = PQ(W_2PQ)^\dagger W_2PQ = PQQ^\dagger(W_2P)^\dagger W_2PQ = PQ = A.$$

In this way, $X \in A\{1,2,4\}$.

On the other hand, consider an arbitrary matrix $X \in A\{1,2,4\}$. Using the general representation of $\{1,2,4\}$-inverses from [**14**], and [**18**], we conclude that X can be represented in the form

$$X = Q^*(QQ^*)^{-1}(YP)^{-1}Y = Q^\dagger(YP)^{-1}Y, \quad Y \in \mathbb{C}_r^{r\times m},$$

where $A = PQ$ is a full-rank factorization of A. Since the conditions (2.1) are satisfied for $U = YP$, $V = Q$, we get

$$X = (YA)^\dagger Y \in \left\{(W_2A)^\dagger W_2 : \quad W_2 \in \mathbb{C}^{r\times m}, \quad W_2P \text{ is invertible}\right\}.$$

(b) It is sufficient to apply the property (2.1) with $U = P$ and $V = QY$, $Y \in \mathbb{C}^{N\times r}$, and the full-rank representation of $\{1,2,3\}$-inverses from [**14**] and [**18**].

(c) This part of the proof follows from the following transformations:

$$\begin{aligned}(P^*A)^\dagger P^* &= Q^\dagger (P^*P)^\dagger P^* Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = A^\dagger,\\ Q^*(AQ^*)^\dagger &= Q^*(QQ^*)^\dagger P^\dagger Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = A^\dagger.\end{aligned}$$

□

In order to compare general representations of {1,2,3} and {1,2,4} inverses with known results [**5**, p. 58] and [**6**, p. 48] we state the following corollary.

Corollary 2.1. *Under the assumptions of Theorem* 2.2 *we have*

$$\begin{aligned}A\{1,2,4\} &= \left\{(W_2A)^\dagger W_2 : \quad W_2 \in \mathbb{C}^{r\times m}, \quad W_2P \textit{ is invertible}\right\}\\ &= \left\{(W_2A)^\dagger W_2 : \quad W_2A \in \mathbb{C}_r^{r\times N}\right\}\\ A\{1,2,3\} &= \left\{W_1(AW_1)^\dagger : \quad W_1 \in \mathbb{C}^{N\times r}, \quad QW_1 \textit{ is invertible}\right\}\\ &= \left\{W_1(AW_1)^\dagger : \quad AW_1 \in \mathbb{C}_r^{m\times r}\right\}.\end{aligned}$$

Remark 2.1. Sharper versions of Proposition 2.1 and Theorem 2.1 are proved in [**10**].

Let $A \in \mathbb{C}_r^{m\times N}$ and $0 < s < r$. Then

$$\begin{aligned}A\{2,4\}_s &= \left\{(W_2A)^{(1,4)}W_2 : \quad W_2A \in \mathbb{C}_s^{s\times N}\right\}\\ A\{2,3\}_s &= \left\{W_1(AW_1)^{(1,3)} : \quad AW_1 \in \mathbb{C}_s^{m\times s}\right\}\\ A\{1,2,4\}_s &= \left\{(W_2A)^{(1,4)}W_2 : \quad W_2A \in \mathbb{C}_r^{r\times N}\right\}\\ A\{1,2,3\}_s &= \left\{W_1(AW_1)^{(1,3)} : \quad AW_1 \in \mathbb{C}_r^{m\times r}\right\}\end{aligned}$$

However, in this paper we use representations from Proposition 2.1 and Theorem 2.1, because the hyper-power method computes the Moore-Penrose inverse.

Introduced representations of generalized inverses are convenient for the application of the modified hyper-power iterative method (1.3). Using this idea, we introduce two modifications of the hyper-power method for construction of $\{1,2,3\}$ and $\{1,2,4\}$ generalized inverses, and two modifications of the hyper-power method for computing subsets of $\{2,3\}$ and $\{2,4\}$-inverses. In these algorithms we consider an arbitrary matrix A of the order $m\times N$. Also, it is assumed that $\operatorname{rank} A = r \geq 2$ and $q \geq 2$ is any integer. *Algorithm* A24 can be used in construction of $\{2,4\}$-inverses, and *Algorithm* A23 can be used in construction of $\{2,3\}$-inverses.

Algorithm A24.

$$
\begin{aligned}
Y_0 &= (W_2A)^*W(W_2A)^*, \text{ for some } W_2 \in \mathbb{C}^{s\times N} \text{ such that}\\
&\quad \rho\left(P_{\mathcal{R}(W_2A)} - W_2AY_0\right) < 1,\ \ 1 < s \le \operatorname{rank}A,\ \ W_2A \in \mathbb{C}_s^{s\times N}\\
X_0 &= Y_0W_2,\\
T_0 &= I - Y_0W_2A,\\
M_k &= I + T_k + {T_k}^2 + \cdots + {T_k}^{q-1},\\
X_{k+1} &= M_kX_k = (I + T_k + {T_k}^2 + \cdots + {T_k}^{q-1})X_k,\\
T_{k+1} &= {T_k}^q = I + M_k[T_k - I],\\
&\quad k = 0, 1, \ldots.
\end{aligned}
\tag{2.1}
$$

This algorithm is an improvement (over using a modification of (1.1) to find $(W_2A)^\dagger$ and then forming $(W_2A)^\dagger W_2$) only in the case $N > m$.

Algorithm A23.

$$
\begin{aligned}
Y_0 &= (AW_1)^*W(AW_1)^*, \text{ for some } W_1 \in \mathbb{C}^{m\times s} \text{ such that}\\
&\quad \rho\left(P_{\mathcal{R}(AW_1)} - AW_1Y_0\right) < 1,\ \ 1 < s \le \operatorname{rank}A,\ \ AW_1 \in \mathbb{C}_s^{m\times s}\\
X_0 &= W_1Y_0,\\
T_0 &= I - AW_1Y_0,\\
M_k &= I + T_k + {T_k}^2 + \ldots + {T_k}^{q-1},\\
X_{k+1} &= X_kM_k = X_k(I + T_k + {T_k}^2 + \cdots + {T_k}^{q-1}),\\
T_{k+1} &= T_k^q = I + M_k[T_k - I],\\
&\quad k = 0, 1, \ldots.
\end{aligned}
\tag{2.3}
$$

This algorithm is an improvement (over using a modification of (1.1) to find $(AW_1)^\dagger$ and then forming $W_1(AW_1)^\dagger$) in the case $m > N$.

Remark 2.2. Instead of the initial approximations Y_0, used in (2.2) and (2.3), we can use the following approximations:

$$Y_0 = \alpha(W_2A)^*, \qquad 0 < \alpha \le \frac{2}{\operatorname{Tr}\left((W_2A)^*W_2A\right)}, \tag{2.2$'$}$$

$$Y_0 = \alpha(AW_1)^*, \qquad 0 < \alpha \le \frac{2}{\operatorname{Tr}\left((AW_1)^*AW_1\right)}. \tag{2.3$'$}$$

Initial approximations (2.2) and (2.3) are more general, but (2.2') and (2.3') are simpler for computation.

Theorem 2.2. *For an arbitrary matrix $A \in \mathbb{C}_r^{m\times N}$, any integer $1 < s \le r$ and arbitrary matrices $W_2 \in \mathbb{C}^{s\times m}$, $W_1 \in \mathbb{C}^{N\times s}$ the following statements are valid:*

(a) *In general, the sequence X_k, $k = 0, 1, \ldots$, defined in Algorithm* A24 *converges to*

$$X_k \to (W_2A)^\dagger W_2 \in A\{2,4\}_s \text{ if and only if } W_2A \in \mathbb{C}_s^{s\times N}. \tag{2.4}$$

(b) *In the case $s=r$ the sequence X_k, $k=0,1,\dots$, defined in Algorithm* A24 *satisfies*

(2.4′) $$X_k \to (W_2A)^\dagger W_2 \in A\{1,2,4\} \text{ if and only if } W_2P \text{ is invertible.}$$

(c) *The optimal order q of methods (a) and (b) minimizes the function*

$$f(q) = (m/N + q - 1)/\ln q.$$

(d) *In general, the sequence X_k, $k=0,1,\dots$, defined in Algorithm* A23 *satisfies*

(2.5) $$X_k \to W_1(AW_1)^\dagger \in A\{2,3\}_s \text{ if and only if } AW_1 \in \mathbb{C}_s^{m\times s}.$$

(e) *In the case $s=r$ the sequence X_k, $k=0,1,\dots$, defined in Algorithm* A23 *satisfies*

(2.5′) $$X_k \to W_1(AW_1)^\dagger \in A\{1,2,3\} \text{ if and only if } QW_1 \text{ is invertible.}$$

(f) *The optimal order q of methods in (d) and (e) minimizes the function*

$$f(q) = (N/m + q - 1)/\ln q.$$

Proof. **(a)**, **(b)** It is not difficult to verify

$$X_k = Y_kW_2, \quad k=0,1,\dots$$

where the sequence $\{Y_k\}$ is defined as in the following:

$$\begin{aligned}
Y_0 &= \alpha(W_2A)^*, \quad 0<\alpha\le \frac{2}{\operatorname{Tr}\big((W_2A)^*W_2A\big)}, \quad W_2A\in\mathbb{C}_s^{s\times N}\\
T_0 &= I - Y_0W_2A,\\
M_k &= I + T_k + T_k^2 + \cdots + T_k^{q-1},\\
Y_{k+1} &= M_kY_k = (I + T_k + T_k^2 + \ldots + T_k^{q-1})Y_k,\\
T_{k+1} &= T_k^q = I + M_k[T_k - I],\\
k &= 0,1,\dots.
\end{aligned}$$

Since the sequence $\{Y_k\}$ is defined by applying the usual hyper-power method (1.1) on the matrix W_2A, we conclude $Y_k \to (W_2A)^\dagger$. Hence, we get $X_k \to (W_2A)^\dagger W_2$. Then statements (2.4) and (2.4') follows from Proposition 2.1 and Theorem 2.1, respectively.

(c) The optimal order q can be determined using the known results from [**8**].

The parts **(d)**, **(e)** and **(f)** can be proved in a similar way. □

In the case rank $A = 1$ the set of $\{1,2,3\}$ and $\{1,2,4\}$-inverses can be generated using the next known proposition from [**17**]:

$$A^{\dagger} = \frac{1}{\mathrm{Tr}\,(A^{*}A)}\,A^{*}.$$

Corollary 2.2. *If A is $m \times N$ matrix satisfying* $\operatorname{rank} A = 1$, *then the following statements are valid:*

(a) $X = W_1(AW_1)^{\dagger} \in A\{1,2,3\}$ *is given by*

$$X = \frac{1}{Tr\big((AW_1)^{*}AW_1\big)}\,W_1(AW_1)^{*}.$$

(b) $Y = (W_2A)^{\dagger}W_2 \in A\{1,2,4\}$ *is given by*

$$Y = \frac{1}{Tr\big((W_2A)^{*}W_2A\big)}\,(W_2A)^{*}W_2.$$

3. IMPLEMENTATION METHOD

In this section we describe implementation details of the introduced algorithms, in the package MATHEMATICA. In the following function norm $[a]$ we compute the norm $\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}$ of the matrix a.

```
norm[a_]:   Block[{m,n,i,j,u},
      {m,n\}=Dimensions[a];
      u=Sum[a[[i,j]]^2,{i,m},{j,n}];
      Return[N[Sqrt[u],20]]
 ]
```

In the function *HyperPower* 124 is implemented *Algorithm* A24.

```
(* A^(2,4)=(W2A)^+ W2 *)
HyperPower124[a_,w2_,q_,eps_]:
         Block[{tk,tk1,c=w2.a,e,x0,x1,y,alpha,s,k=1,m,n,nor=1},
         alpha=2/trace[Transpose[Conjugate[c]].c];
         y=alpha Conjugate[Transpose[c]];
         x0=y.w2;
         e=IdentityMatrix[n];
         tk=s=e-y.c;
         While[nor>=eps,
               s=e;  tk1=tk; Do[s+=tk1; tk1=tk1.tk,{i,q-1}];
               x1=s.x0;
```

```
              tk=tk1;
              nor=norm[x1-x0];
              x0=x1;    k=k+1
             ];
        N[x1]
  ]
```

In the function *HyperPower* 123 is implemented *Algorithm* A23.

```
(* A^ {}(2,3)=W1(AW1)^ {}+ *)
HyperPower123[a_,w1_,q_,eps_]:
        Block[{tk,tk1,c=a.w1,e,x0,x1,y,alpha,s,k=1,m,n,nor=1},
        alpha=2/trace[Transpose[Conjugate[c]].c];
        y=alpha Conjugate[Transpose[c]];
        x0=w1.y;
        e=IdentityMatrix[n];
        tk=s=e-c.y;
        While[nor>=eps,
              s=e;  tk1=tk; Do[s+=tk1; tk1=tk1.tk,\{i,q-1\}];
              x1=x0.s;
              tk=tk1;
              nor=norm[x1-x0];
              x0=x1;     k=k+1
             ];
        N[x1]
    ]
```

Example 3.1. In this example we construct $\{1,2,4\}$ and $\{1,2,3\}$-inverse of the matrix

$$A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -2 \end{bmatrix}$$

which are generated, respectively, by the matrices

$$W_2 = \begin{bmatrix} 3 & 1 & 3 & 1 & 2 & -1 \\ 0 & -1 & 0 & 0 & -2 & 1 \end{bmatrix}, \quad W_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \\ 4 & 2 \end{bmatrix}. \tag{3.1}$$

By means of the expression $HyperPower124[a, w2, 3, 10\hat{\ }(-11)]$ we apply the *Algorithm* A24 for the matrices A, W_2, using the order $q = 3$ and the precision 10^{-11}. In this case is $s = \operatorname{rank} A = 2$, and the resulting matrix

$$\begin{bmatrix} -0.647059 & 0.843137 & -0.647059 & -0.215686 & 1.68627 & -0.843137 \\ 0.411765 & -0.627451 & 0.411765 & 0.137255 & -1.2549 & 0.627451 \\ 0.235294 & -0.215686 & 0.235294 & 0.0784314 & -0.431373 & 0.215686 \\ 0.0588235 & 0.196078 & 0.0588235 & 0.0196078 & 0.392157 & -0.196078 \end{bmatrix}$$

is an $\{1,2,4\}$-inverse of A.

Similarly, by means of the expression $\mathit{HyperPower}123[a, w1, 3, 10\hat{}(-11)]$ we apply the *Algorithm* A23 for the matrices A, W_1, using the modified Hyper-power method of the order 3, with the precision 10^{-11}. The resulting $\{1,2,3\}$-inverse of A is equal to

$$\begin{bmatrix} -0.117647 & -0.176471 & 0.0588235 & -0.0588235 & 0.176471 & 0.117647 \\ 0.186275 & 0.196078 & -0.00980392 & 0.00980392 & -0.196078 & -0.186275 \\ -0.0588235 & -0.08882353 & 0.0294118 & -0.0294118 & 0.0882353 & 0.0588235 \\ 0.137255 & 0.0392157 & 0.0980392 & -0.0980392 & -0.0392157 & -0.137255 \end{bmatrix}.$$

Consider now the following matrix of rank 3

$$A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ -1 & 3 & 0 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -2 \end{bmatrix}$$

and the matrices W_1, W_2 as in (3.1). In this case is $s = 2 < \operatorname{rank} A = 3$, and the expression $\mathit{HyperPower124}[a, w2, 4, 10\hat{}(-12)]$ produces the following $\{2,4\}$-inverse of A :

$$\begin{bmatrix} -0.205821 & 0.243243 & -0.205821 & -0.0686071 & 0.486486 & -0.243243 \\ -0.380457, & 0.540541 & -0.380457 & -0.126819 & 1.08108 & -0.540541 \\ 0.0997921 & -0.027027 & 0.0997921 & 0.033264 & -0.0540541 & 0.027027 \\ 0.0935551 & 0.162162 & 0.0935551 & 0.031185 & 0.324324 & -0.162162 \end{bmatrix}.$$

Also, the result of the expression $\mathit{HyperPower}123[a, w1, 4, 10\hat{}(-12)]$ is the following $\{2,3\}$-inverse of A:

$$\begin{bmatrix} -0.038633 & -0.170877 & 0.0401189, & -0.0401189 & 0.0787519 & 0.038633 \\ 0.0903913 & 0.20109 & 0.0151065 & -0.0151065 & -0.0752848 & -0.0903913 \\ -0.0193165 & -0.0854383 & 0.0200594 & -0.0200594 & 0.0393759 & 0.0193165 \\ 0.103517 & 0.060426 & 0.110451 & -0.110451 & 0.00693413 & -0.103517 \end{bmatrix}.$$

4. CONCLUSION

In this section we present a few concluding remarks and comparisons of the introduced method with the modification of the hyper-power method introduced in [**17**].

Remark 4.1. We point out the following advantages of defined algorithms with respect to modifications of the hyper-power method which are introduced in [**17**]:

(a) In the iterations defined in this paper it is not necessary to multiply the matrix M_kX_k (or the matrix X_kM_k) by the matrices W_1 and W_2 from the left and right, respectively.

(b) Iterations defined in [**17**] require a full-rank factorization of A in computation of $\{1,2,3\}$ and $\{1,2,4\}$-inverses. On the other hand, the iterations (2.2) and (2.3) do not require the full-rank factorization.

In order to demonstrate these advantages we consider, in parallel, iterations introduced in this paper and in [**17**].

By means of *Algorithm* A24, the sets $A\{2,4\}_s$ and $A\{1,2,4\}$ can be generated as follows:

$$\begin{aligned}
Y_0 &= \alpha(W_2A)^*, \quad 0<\alpha\leq \frac{2}{\operatorname{Tr}\left((W_2A)^*W_2A\right)}\,, \quad W_2P \text{ is invertible},\\
X_0 &= Y_0W_2, \quad T_0 = I - Y_0W_2A,\\
X_{k+1} &= (I+T_k+T_k^2+\cdots+T_k^{q-1})X_k,\\
T_{k+1} &= T_k^q, \quad k = 1,\dots .
\end{aligned}$$

Applying iterations from [**17**], Lemma 2.1], we must compute a full-rank factorization $A=PQ$ and then generate the following iterations:

$$\begin{aligned}
Y_0 &= \alpha(W_2AQ^*)^*, \qquad 0<\alpha\leq \frac{2}{\operatorname{Tr}\left((W_2AQ^*)^*W_2AQ^*\right)}\\
T_k &= I - Y_kW_2AQ^* = T_{k-1}^q\\
Y_{k+1} &= (I+T_k+T_k^2+\cdots+T_k^{q-1})Y_k,\\
X_{k+1} &= W_1Y_{k+1}W_2, \quad k=0,1,\dots .
\end{aligned}$$

Similarly, by means of *Algorithm* A23, the set $A\{1,2,3\}$ can be generated as follows:

$$\begin{aligned}
Y_0 &= \alpha(AW_1)^*, \quad 0<\alpha\leq \frac{2}{\operatorname{Tr}\left((AW_1)^*AW_1\right)}\,, \quad QW_1 \text{ is invertible},\\
X_0 &= W_1Y_0, \quad T_0 = I - AW_1Y_0,\\
X_{k+1} &= X_k(I+T_k+T_k^2+\cdots+T_k^{q-1}),\\
T_{k+1} &= T_k^q, \quad k=0,1,\dots .
\end{aligned}$$

Also, using the method defined in [**17**, Lemma 2.1], we must compute a full-rank factorization $A = PQ$ and generate the following iterations:

$$\begin{aligned} Y_0 &= \alpha(P^*AW_1)^*, \qquad 0 < \alpha \le \frac{2}{\mathrm{Tr}\left(((P^*AW_1)^*P^*AW_1\right)} \\ T_k &= I - P^*AW_1Y_k = T_{k-1}^q \\ Y_{k+1} &= Y_k(I + T_k + T_k^2 + \cdots + T_k^{q-1}), \\ X_{k+1} &= W_1Y_{k+1}W_2, \quad k = 0, 1, \ldots . \end{aligned}$$

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