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AN INTEGRAL INEQUALITY FOR NON-NEGATIVE POLYNOMIALS

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If p is a polynomial of the exact degree n, $p \geq 0$ on $[a,b]$, the aim of this paper is to establish an inequality of the form

$$\frac{1}{b-a}\int_a^b p(x)\,\mathrm{d}x \geq \alpha_n\big(p(a)+p(b)\big)+\beta_n\,|p(a)-p(b)|.$$

There are given all extremal polynomials for which the equality case is attained.

1. Supposing that n is a fixed natural number, let us define

$$m=\left[\frac{n}{2}\right],\quad s=n-2m,\quad d=\left[\frac{n+1}{2}\right], \tag{1}$$

where the symbol $[\cdot]$ denotes the integral part.

By Π we denote the algebra of univariate polynomials with real coefficients and by Π_n the real linear space of all polynomials from Π of the degree at most n.

The set ${\mathcal{P}_n}^+[a,b]$ consists of the polynomials p, $p \in \Pi_n$, with the properties

$$\text{degree}\,[p]=n,\quad p(x)\geq 0 \text{ for all } x\in[a,b].$$

Let

$$R_n^{(\alpha,\beta)}(x)={}_2F_1\left(-n,n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\right)=\frac{P_n^{(\alpha,\beta)}(x)}{\binom{n+\alpha}{n}}$$

be the JACOBI polynomial of the degree n normalized by $R_n^{(\alpha,\beta)}(1)=1$.

Further, for a fixed n, let us select the division

$$-1<z_1<z_2<\cdots<z_m<1,$$

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(2) $$R_m{}^{(1,s)}(z_j) = 0, \qquad j = 1, 2, \ldots, m,$$

m and s having the same meaning as in (1). At the same time we shall denote $y_k = -z_{m+1-k}$, $k = 1, 2, \ldots, m$. It is known that

(3) $$R_m^{(s,1)}(y_k) = 0, \qquad k = 1, 2, \ldots, m.$$

Suppose that m, s, d, z_k are as in (1)–(2) and let us denote

$$C_0 = \frac{8s}{(n+2)^2 - s}\,,\ C_{m+1} = \frac{8}{(n+2)^2 - s}\,,\ G_k = \frac{1 + (1-s)z_k}{\Big(R_d{}^{(0,0)}(z_k)\Big)^2}\,,\ k = 1, \ldots, m,$$

$$\zeta = \frac{2^{s+2}}{(n+2)^2 - s}\,, \quad \eta_n = \frac{2^{n+2} m!\,(m+1)!d!\,(d+1)!}{(n+1)!\,(n+2)!}\,, \quad \mu_n = (-1)^{n+1}\eta_n.$$

Further we need the following result.

Lemma 1. *If $f \in C^{n+1}[-1,1]$, then there exist in $(-1,1)$ at least two points θ_1, θ_2 such that*

(4) $$\int_{-1}^{1} f(x)\,\mathrm{d}x = C_0 f(-1) + \zeta \sum_{k=1}^{m} G_k\, f(z_k) + C_{m+1} f(1) - \eta_n\, \frac{f^{(n+1)}(\theta_1)}{(n+1)!}\,,$$

(5) $$\int_{-1}^{1} f(x)\,\mathrm{d}x = C_0 f(1) + \zeta \sum_{k=1}^{m} G_{m+1-k}\, f(y_k) + C_{m+1} f(-1) - \mu_n\, \frac{f^{(n+1)}(\theta_2)}{(n+1)!}\,.$$

Proof. In order to justify (4)–(5) it is sufficient to use the Bouzitat quadrature formulas for the Legendre weight, i.e. the Radau formulas. Let $n = 2m + s$. When n is even ($s = 0$) the quadrature from (5) may be derived, by a suitable linear transformation, from the Bouzitat formula of the first kind (see [**2**]). If $n = 2m + 1$, then (4) is the same with Bouzitat elementary quadrature formulae of the second kind (see (4.7.1) and (4.8.1) from [**2**]) .

2. Let us consider the polynomials H^* and G^* where

$$H^*(x) := \big(\mathcal{A}(x-a)^s + \mathcal{B}s(b-x)\big)\bigg(R_m^{(1,s)}\Big(\frac{2x-a-b}{b-a}\Big)\bigg)^2,$$
$$G^*(x) := H^*(a+b-x),$$

where $\mathcal{A} > 0$ when $s = 0$ and $\mathcal{A} \geq 0$, $\mathcal{B} \geq 0$, $\mathcal{A} \neq \mathcal{B}$ for $s = 1$.

The main result is the following proposition:

Theorem 1. *If $p \in \mathcal{P}_n{}^{+}[a,b]$, $m = [n/2]$, $s = n - 2m$, then*

$$\frac{1}{b-a}\int_a^b p(x)\,\mathrm{d}x \geq \frac{2(1+s)}{(n+2)^2 - s}\,\big(p(a) + p(b)\big) + \frac{2(1-s)}{(n+2)^2 - s}\,\big|p(a) - p(b)\big|.$$

The equality case is attained only for the polynomials $H^*(x)$ or $G^*(x)$.

Proof. First, let us observe that the extremal polynomial H^* may be written as

$$H^*(x) = \frac{\mathcal{A}(x-a)^s + \mathcal{B}s\,(b-x)}{(x-a)^{2s}} \left(\frac{P_m\big(y(x)\big) - P_{m+1+s}\big(y(x)\big)}{b-x} \right)^2.$$

where $y(x) = \dfrac{2x-a-b}{b-a}$ and $P_m(x) = \dfrac{1}{2^m\, m!}\big((x^2-1)^m\big)^{(m)}$ being the LEGENDRE polynomial.

We shall prove the above result in the case $[a,b]=[-1,1]$. Supposing that

$$E_n(p;s) := \frac{4(1+s)}{(n+2)^2-s}\big(p(-1)+p(1)\big) + \frac{4(1-s)}{(n+2)^2-s}\big|p(-1)-p(1)\big|,$$

we must prove that the inequality

$$\int_{-1}^{1} p(x)\,\mathrm{d}x \geq E_n(p;s) \tag{6}$$

is verified for any polynomial p, of the degree n, which is non-negative on the interval $[-1,1]$.

Using the notation (1) we observe that if $p \in \mathcal{P}_n{}^+[-1,1]$, then there exist two polynomials A_m, B_{d-1}

$$A_m \in \Pi_m, \qquad A_m(1) = p(1), \qquad B_{d-1} \in \Pi_{d-1},$$

such that

$$p(x) = \left(\frac{1+x}{2}\right)^s A_m{}^2(x) + (1-x)(1+x)^{1-s} B_{d-1}{}^2(x). \tag{7}$$

This representation is in fact a form of a theorem of LUKÁCS (see [**4**]–[**5**]). According to (4)

$$\begin{aligned}\int_{-1}^{1} p(x)\,\mathrm{d}x &= \frac{8}{(n+2)^2-s}\big(sp(-1)+p(1)\big) + \zeta \sum_{k=1}^{m} G_k\, p(z_k)\\ &\geq \frac{8}{(n+2)^2-s}\big(sp(-1)+p(1)\big)\end{aligned}$$

and in the same manner from (5)

$$\begin{aligned}\int_{-1}^{1} p(x)\,\mathrm{d}x &= \frac{8}{(n+2)^2-s}\big(p(-1)+sp(1)\big) + \zeta \sum_{k=1}^{m} G_{m+1-k}\, p(y_k)\\ &\geq \frac{8}{(n+2)^2-s}\big(p(-1)+sp(1)\big).\end{aligned}$$

For n even ($s=0$), from the above inequalities we find

$$\int_{-1}^{1} p(x)\,\mathrm{d}x \geq \frac{8}{(n+2)^2}\ \max\{p(-1),p(1)\}E_n(p;0).$$

When $s=1$, that is, n is odd one observes that

$$\int_{-1}^{1} p(x)\,\mathrm{d}x \geq \frac{8}{(n+2)^2-1}\,\big(p(-1)+p(1)\big)\,E_n(p;1).$$

Let us show that (6) cannot be improved. We shall show that there exists a polynomial $h^* \in \mathcal{P}_n{}^+[-1,1]$ such that

$$\int_{-1}^{1} h^*(x)\,\mathrm{d}x = E_n(h^*;s).$$

Consider the polynomial

$$h^*(x) = \Big(\Big(\frac{1+x}{2}\Big)^s + Bs\Big(\frac{1-x}{2}\Big)\Big)\Big(R_m^{(1,s)}(x)\Big)^2, \tag{8}$$

where $B \geq 0$, $B \neq 1$. It is clear that $h^* \in \mathcal{P}_n{}^+$, and moreover

$$h^*(-1) = \frac{4(1-s)}{(n+1)^2} + sB, \quad h^*(1) = 1, \quad h^*(z_k) = 0, \quad k = 1,2,\ldots,m.$$

At the same time, according to (4)–(5)

$$\int_{-1}^{1} h_1^*(x)\,\mathrm{d}x = \frac{8\,\big(1+sh^*(-1)\big)}{(n+2)^2-s} = E_n(h^*;s)$$

which means that the equality case in (6) holds.

Let us investigate the equality cases:

(i) $s=0$: then $d=m$ and $p(x) = A_m^2(x) + (1-x^2)B_{m-1}^2(x)$.

In order to have equality in (7) we must have one of the following two situations:

a) $B_{m-1}(z_k)=0, \quad A_m(z_k)=0, \quad k=1,2,\ldots,m$, and $p(1) \geq p(-1)$, or

b) $B_{m-1}(y_k)=0, \quad A_m(y_k)=0, \quad k=1,2,\ldots,m$, and $p(-1) \geq p(1)$.

When *a)* holds, then $B_{m-1}=0$ and

$$p(x) = \lambda_1\Big(R_m^{(1,0)}(x)\Big)^2, \quad \lambda_1 > 0,$$

that is $p(x) = \lambda_1 h^*(x)$.

Suppose that b) is true. Then we observe that

$$p(x) = \lambda_2 \Big(R_m^{(0,1)}(x) \Big)^2 = \lambda_3 \Big(R_m^{(1,0)}(-x) \Big)^2, \ \ \lambda_2, \lambda_3 > 0.$$

Thus, $p(x)$ must be of the form $p(x) = \lambda_4 h^*(-x),\ \lambda_4 > 0$.

(ii) $s = 1$: then $d - 1 = m$ and

$$p(x) = \frac{1+x}{2}\, A_m^2(x) + (1-x)\, B_{m-1}^2(x).$$

It may be seen that for equality case it is necessary and sufficient to have

$$p(z_k) = 0, \qquad k = 1, 2, \ldots, m,$$

therefore $B_{d-1}(x) = \mu_1 R_m^{(1,1)}(x)$, $B_m(x) = \mu_2 R_m^{(1,1)}(x)$. In other words, the extremal polynomials $p^*(x)$ have the representation

$$p^*(x) = (\mu x + \nu) \left(R_m^{(1,1)}(x) \right)^2 = \mu_1 h^*(x) = \mu_2 h^*(-x)$$

with $0 < |\mu| \leq \nu$, $\mu_1 > 0$, $\mu_2 > 0$.

In conclusion, the equality in (6) holds only for polynomials from ${\mathcal{P}_n}^+[-1,1]$ which have the representation

$$p(x) = \psi_1 h^*(x) \ \text{ or } \ p(x) = \psi_2 h^*(-x),$$

ψ_1, ψ_2 being positive constants and $h^*(x)$ as in (8).

REMARK. When n is even, i.e. $s = 0$ the above theorem was proved by F. LUKÁCS in [**3**]: see also the excellent monograph [**4**], pages 132–133 (Theorem 1.7.1 and Theorem 1.7.2).

3. A more general problem may be formulated in the following manner. We consider a non-negative weight $w(x)$ which is defined on a bounded interval $[a, b]$ such that all moments

$$\int_a^b x^i w(x)\,dx, \quad i = 0, 1, \ldots, \qquad \int_a^b w(x)\,dx = 1$$

are finite and $\operatorname{supp}(w)$ has a positive measure.

Suppose that z_1, z_2 are fixed points

$$-\infty < a \leq z_1 < z_2 \leq b < \infty,\ (z_1 - a)^2 + (b - z_2)^2 > 0.$$

Problem. *Find the "best constants"*

$$K_{1,n} = K_{1,n}(z_1, z_2; w),\ K_{2,n} = K_{2,n}(z_1, z_2; w)$$

in order that the inequality

$$\int_a^b p(x)w(x)\,\mathrm{d}x \geq K_{1,n}\big(p(z_1)+p(z_2)\big)+K_{2,n}\big|p(z_1)-p(z_2)\big|,$$

be valid for all $p \in {\mathcal{P}_n}^+[a,b]$.

It seems that in the case $a = z_1 < z_2 < b$, or when $a < z_1 < z_2 = b$ the above problem may be solved (n odd) by means of FILLIPPI quadrature formula (see [**1**], p. 328).

REFERENCES

1. H. ENGELS: *Numerical quadrature and cubature.* Academic Press, New York, 1980.
2. A. GHIZZETTI, A. OSSICINI: *Quadrature formulae.* Birkhäuser Verlag , ISNM vol. **13**, Basel, 1970.
3. F. LUKÁCS: *Verschärfung des ersten Mittelwertsatzes der Integralrechnung für rationale Polynome.* Math. Z. **2** (1918), 295–305.
4. G. V. MILOVANOVIĆ , D. S. MITRINOVIĆ , TH.M.RASSIAS: *Topics in polynomials: Extremal problems, Inequalities, Zeros.* World Scientific, Singapore, London, 1994.
5. G.SZEGÖ: *Orthogonal polynomials.* Amer. Math. Soc. Colloq. Publications, **23** (1978).

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