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SOME IDENTITIES FOR THE RIEMANN ZETA-FUNCTION

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Several identities for the Riemann zeta-function $\zeta(s)$ are proved. For example, if $s = \sigma + it$ and $\sigma > 0$, then

$$\int_{-\infty}^{\infty} \left| \frac{(1-2^{1-s})\zeta(s)}{s} \right|^2 \mathrm{d}t = \frac{\pi}{\sigma}(1-2^{1-2\sigma})\zeta(2\sigma).$$

Let as usual $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ ($\Re\mathrm{e}\, s > 1$) denote the Riemann zeta-function. The motivation for this note is the quest to evaluate explicitly integrals of $|\zeta(\frac{1}{2} + it)|^{2k}$, $k \in \mathbb{N}$, weighted by suitable functions. In particular, the problem is to evaluate in closed form

$$\int_0^{\infty} \bigl(3 - \sqrt{8}\cos(t\log 2)\bigr)^k |\zeta(\tfrac{1}{2} + it)|^{2k} \frac{\mathrm{d}t}{\bigl(\frac{1}{4} + t^2\bigr)^k} \qquad (k \in \mathbb{N}).$$

When $k = 1, 2$ this may be done, thanks to the identities which will be established below. The first identity in question is given by

Theorem 1. *Let $s = \sigma + it$. Then for $\sigma > 0$ we have*

$$\int_{-\infty}^{\infty} \left| \frac{(1-2^{1-s})\zeta(s)}{s} \right|^2 \mathrm{d}t = \frac{\pi}{\sigma}(1-2^{1-2\sigma})\,\zeta(2\sigma). \tag{1}$$

Since $\lim_{s\to 1}(s-1)\zeta(s) = 1$, then setting in (1) $\sigma = \frac{1}{2}$ we obtain the following

Corollary 1.

$$\int_0^{\infty} \bigl(3 - \sqrt{8}\cos(t\log 2)\bigr)|\zeta(\tfrac{1}{2} + it)|^2 \frac{\mathrm{d}t}{\frac{1}{4} + t^2} \;=\; \pi \log 2. \tag{2}$$

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Another identity, which relates directly the square of $\zeta(s)$ to a Mellin-type integral, is contained in

Theorem 2. *Let $\chi_{\mathcal{A}}(x)$ denote the characteristic function of the set $\mathcal{A}$, and let*

$$\varphi(x) := \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_1^x \chi_{[2m-1,2m)}\Big(\frac{x}{u}\Big)\chi_{[2n-1,2n)}(u)\,\frac{\mathrm{d}u}{u} \qquad (x \geqslant 1). \tag{3}$$

Then for $\sigma > 0$ we have

$$s^2\int_1^{\infty}\varphi(x)x^{-s-1}\,\mathrm{d}x = (1-2^{1-s})^2\zeta^2(s). \tag{4}$$

From (4) we obtain the following

Corollary 2.

$$\int_0^{\infty}\big(3-\sqrt{8}\cos(t\log 2)\big)^2\big|\zeta(\tfrac12+it)\big|^4\frac{\mathrm{d}t}{\big(\frac14+t^2\big)^2} \;=\; \pi\int_1^{\infty}\varphi^2(x)\frac{\mathrm{d}x}{x^2}. \tag{5}$$

The integral on the right-hand side of (5) is elementary, but nevertheless its evaluation in closed form is complicated.

Proof of Theorem 1. We start from (see e.g., [**1**, Chapter 1]) the identity

$$(1-2^{1-s})\zeta(s) \;=\; \sum_{n=1}^{\infty}(-1)^{n-1}n^{-s} \qquad (\sigma > 0) \tag{6}$$

and

$$\int_{-\infty}^{\infty}\frac{\cos(\alpha x)}{\sigma^2+x^2}\,\mathrm{d}x \;=\; \frac{\pi}{\sigma}\mathrm{e}^{-|\alpha|\sigma} \qquad (\alpha\in\mathbb{R},\ \sigma>0), \tag{7}$$

which follows by the residue theorem on integrating $e^{i\alpha z}/(\sigma^2+z^2)$ over the contour consisting of $[-R,\,R]$ and semicircle $|z| = R, \Im\mathrm{m}\, z > 0$ and letting $R\to\infty$. By using (6) and (7) it is seen that the left-hand side of (1) becomes

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}(mn)^{-\sigma}\int_{-\infty}^{\infty}\Big(\frac{m}{n}\Big)^{it}\frac{\mathrm{d}t}{\sigma^2+t^2}\\
&\quad= \frac{\pi}{\sigma}\zeta(2\sigma) + 2\sum_{m=1}^{\infty}\sum_{n<m}(-1)^{m+n}(mn)^{-\sigma}\int_{-\infty}^{\infty}\frac{\cos(t\log\frac{m}{n})}{\sigma^2+t^2}\,\mathrm{d}t\\
&\quad= \frac{\pi}{\sigma}\left(\zeta(2\sigma) + 2\sum_{m=1}^{\infty}(-1)^m m^{-\sigma}\sum_{n=1}^{m-1}(-1)^n n^{-\sigma}\cdot e^{-\sigma\log\frac{m}{n}}\right)\\
&\quad= \frac{\pi}{\sigma}\left(\zeta(2\sigma) + 2\sum_{m=1}^{\infty}(-1)^m m^{-2\sigma}\sum_{n=1}^{m-1}(-1)^n\right)\\
&\quad= \frac{\pi}{\sigma}\left(\zeta(2\sigma) + 2\sum_{k=1}^{\infty}(-1)^{2k}(2k)^{-2\sigma}(-1)\right) = \frac{\pi}{\sigma}(1-2^{1-2\sigma})\zeta(2\sigma).
\end{aligned}$$

This holds initially for $\sigma > 1$, but by analytic continuation it holds for $\sigma > 0$ as well.

We shall provide now a second proof of Theorem 1. As in the formulation of Theorem 2, let $\chi_{\mathcal{A}}(x)$ denote the characteristic function of the set $\mathcal{A}$, and let the interval $[a, b)$ denote the set of numbers $\{x : a \leqslant x < b\}$. Then, for $\sigma > 0$, we have

$$\begin{aligned}\int_1^\infty x^{-s-1}\sum_{n=1}^\infty \chi_{[2n-1,2n)}(x)\,\mathrm{d}x &= \sum_{n=1}^\infty \int_{2n-1}^{2n} x^{-s-1}\,\mathrm{d}x\\ = \frac{1}{s}\sum_{n=1}^\infty \big((2n-1)^{-s} - (2n)^{-s}\big) &= \frac{(1-2^{1-s})\,\zeta(s)}{s}\end{aligned}\tag{8}$$

in view of (6). Now we invoke Parseval's identity for Mellin transforms (see e.g., [**1**] and [**3**]). We need this identity for the modified Mellin transforms, defined by

$$F^*(s) \equiv m[f(x)] := \int_1^\infty f(x)\,x^{-s-1}\,\mathrm{d}x.$$

The properties of this transform were developed by the author in [**2**]. In particular, we need Lemma 3 of [**2**] which says that

$$\int_1^\infty f(x)\,g(x)\,x^{1-2\sigma}\,\mathrm{d}x = \frac{1}{2\pi i}\int_{\Re\mathrm{e}\, s=\sigma} F^*(s)\,\overline{G^*(s)}\,\mathrm{d}s\tag{9}$$

if $F^*(s) = m[f(x)]$, $G^*(s) = m[g(x)]$, and $f(x), g(x)$ are real-valued, continuous functions for $x > 1$, such that

$$x^{\frac12-\sigma}f(x) \in L^2(1,\infty), \quad x^{\frac12-\sigma}g(x) \in L^2(1,\infty).$$

From (8) and (9) we obtain, for $\sigma > 0$,

$$\int_1^\infty \frac{1}{x^2}\left(\sum_{n=1}^\infty \chi_{[2n-1,2n)}(x)\right)^2 x^{1-2\sigma}\,\mathrm{d}x = \frac{1}{2\pi i}\int_{\Re\mathrm{e}\, s=\sigma}\left|\frac{(1-2^{1-s})\zeta(s)}{s}\right|^2\mathrm{d}s.$$

But as $\chi_{\mathcal{A}}^2(x) = \chi_{\mathcal{A}}(x)$, it is easily found that the left-hand side of the above identity equals

$$\begin{aligned}&\sum_{m=1}^\infty\sum_{n=1}^\infty\int_1^\infty \chi_{[2m-1,2m)}(x)\chi_{[2n-1,2n)}(x)x^{-1-2\sigma}\,\mathrm{d}x\\ &= \sum_{n=1}^\infty\int_{2n-1}^{2n} x^{-1-2\sigma}\,\mathrm{d}x = \frac{(1-2^{1-2\sigma})\zeta(2\sigma)}{2\sigma}\end{aligned}$$

in view of (6), and (1) follows. □

For the Proof of Theorem 2 we need the following

Lemma. *Let* $0 < a < b$. *If* $f(x)$ *is integrable on* $[a, b]$, *then*

$$\begin{aligned}(10)\qquad &\left(\int_a^b f(x)x^{-s}\,\mathrm{d}x\right)^2\\ &= \int_{a^2}^{ab} x^{-s}\int_a^{x/a} f(u)f\Big(\frac{x}{u}\Big)\frac{\mathrm{d}u}{u}\,\mathrm{d}x + \int_{ab}^{b^2} x^{-s}\int_{x/b}^{b} f(u)f\Big(\frac{x}{u}\Big)\frac{\mathrm{d}u}{u}\,\mathrm{d}x.\end{aligned}$$

The identity (10) *remains valid if* $b = \infty$, *provided the integrals in question converge, in which case the second integral on the right-hand side is to be omitted.*

Proof. We write the left-hand side of (10) as the double integral

$$\int_a^b\int_a^b (xy)^{-s}f(x)f(y)\,\mathrm{d}x\,\mathrm{d}y$$

and make the change of variables $x = X/Y, y = Y$. The Jacobian of this transformation equals $1/Y$, hence the left-hand side of (10) becomes

$$\begin{aligned}&\int_{a^2}^{b^2} X^{-s}\left(\int_{\max(a,X/b)}^{\min(X/a,b)} f(Y)f\Big(\frac{X}{Y}\Big)\frac{\mathrm{d}Y}{Y}\right)\mathrm{d}X\\ &= \int_{a^2}^{ab} X^{-s}\int_a^{X/a} f(Y)f\Big(\frac{X}{Y}\Big)\frac{\mathrm{d}Y}{Y}\,\mathrm{d}X + \int_{a^b}^{b^2} X^{-s}\int_{X/b}^{b} f(Y)f\Big(\frac{X}{Y}\Big)\frac{\mathrm{d}Y}{Y}\,\mathrm{d}X,\end{aligned}$$

as asserted.

Proof of Theorem 2. We use (8) and the Lemma to obtain that (4) certainly holds with $\varphi(x)$ given by (3), since trivially $\varphi(x) \ll x$. To see that it holds for $\sigma > 0$, we note that

$$(11)\qquad \int_1^x g(u)g\Big(\frac{x}{u}\Big)\frac{\mathrm{d}u}{u} = \int_1^{\sqrt{x}} + \int_{\sqrt{x}}^x = 2\int_{\sqrt{x}}^x g(u)\,g\Big(\frac{x}{u}\Big)\,\frac{\mathrm{d}u}{u},$$

and use (11) with

$$g(x) = \sum_{n=1}^{\infty}\chi_{[2n-1,2n)}(x).$$

Note then that the integrand in $\varphi(x)$ equals $1/u$ for $2m-1 \leqslant u \leqslant 2m$, $2n-1 \leqslant u \leqslant 2n$, and otherwise it is zero. This gives the condition

$$4mn - 2m - 2n + 1 \leqslant x < 4mn,\ \tfrac{1}{2}\sqrt{x} \leqslant n \leq \tfrac{1}{2}(x+1),\ 1 \leqslant m \leqslant \tfrac{1}{2}(\sqrt{x}+1).$$

We also have

$$\int_{\sqrt{x}}^x \chi_{[2m-1,2m)}\big(\frac{x}{u}\big)\chi_{[2n-1,2n)}(u)\frac{\mathrm{d}u}{u} \leqslant \int_{2n-1}^{2n}\frac{\mathrm{d}u}{u} \leqslant \frac{1}{2n-1}.$$

Therefore

$$
\begin{aligned}
\varphi(x) &\ll \sum_{m\leqslant\sqrt{x}} \; \sum_{x/(4m)<n\leqslant(x-1+2m)/(4m-2)} \frac{1}{n} \\
&\ll \sum_{m\leqslant\sqrt{x}} \frac{m}{x}\left(1+\frac{x}{m^2}\right) \ll \log x.
\end{aligned} \tag{12}
$$

This bound shows that the integral in (4) is absolutely convergent for $\sigma > 0$. Thus by the principle of analytic continuation this completes the proof of Theorem 2. □

Corollary 2 follows then from (4) and (9) on setting $\sigma = \frac{1}{2}$.

It is interesting to note that the bound in (12) is actually of the correct order of magnitude. Namely we have

Theorem 3. *For any given $\varepsilon > 0$ we have*

$$
\varphi(x) \;=\; \tfrac{1}{4}\log x + \tfrac{1}{2}\log\left(\tfrac{\pi}{2}\right) + O_\varepsilon\left(x^{\varepsilon-\frac{1}{4}}\right). \tag{13}
$$

Proof. By (8) and the inversion formula for the MELLIN transform $m[f(x)]$ (see [2, Lemma 1]) we have, for any $c > 0$,

$$
\varphi(x) = \frac{1}{2\pi i}\int_{\Re\mathrm{e}\, s=c} \frac{(1-2^{1-s})^2\zeta^2(s)x^s}{s^2}\,\mathrm{d}s. \tag{14}
$$

We shift the line of integration in (14) to $c = \varepsilon - 1/4$ with $0 < \varepsilon < 1/8$, which clearly may be assumed. Since $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, the residue at the double pole $s = 0$ is found to be

$$
\frac{1}{4}\log x + A, \quad A = -\zeta'(0) - \log 2 = \tfrac{1}{2}\log\left(\frac{\pi}{2}\right). \tag{15}
$$

We use the functional equation (see e.g., [**1**, Chapter 1]) for $\zeta(s)$, namely

$$
\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = 2^s\pi^{s-1}\sin(\tfrac{1}{2}\pi s)\Gamma(1-s)
$$

with

$$
\chi(s) = \left(\frac{2\pi}{t}\right)^{\sigma+it-\frac{1}{2}} \mathrm{e}^{i(t+\frac{1}{4}\pi)}\cdot\left(1+O\Big(\frac{1}{t}\Big)\right) \quad (t \geqslant 2).
$$

Let $s = \varepsilon - \frac{1}{4} + it$. Then by absolute convergence we have

$$
\begin{aligned}
&\int_T^{2T} \frac{(1-2^{1-s})^2\zeta^2(s)x^s}{s^2}\,\mathrm{d}t \\
&= i\sum_{n=1}^\infty d(n)n^{\varepsilon-5/4}\int_T^{2T} \frac{(1-2^{1-s})^2}{s^2}x^{\varepsilon-\frac{1}{4}+it}\left(\frac{t}{2\pi}\right)^{\frac{3}{2}-2\varepsilon}\mathrm{e}^{iF(t,n)}\,\mathrm{d}t + O(T^{-\frac{1}{2}-2\varepsilon}),
\end{aligned}
$$

where $d(n)$ is the number of divisors of n and

$$F(t,n) := 2t + t\log n - 2t\log(t/2\pi), \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\,(t\log x + F(t,n)) = -\frac{2}{t}\,.$$

Hence by the second derivative test (see [**1**, Lemma 2.2]) the above series is

$$\ll \sum_{n=1}^{\infty} d(n)n^{\varepsilon-5/4}T^{-2\varepsilon} = \zeta^2\big(\tfrac{5}{4} - 2\varepsilon\big)T^{-2\varepsilon} \ll T^{-2\varepsilon}.$$

This shows that

$$\int_{\Re\mathrm{e}\, s=\varepsilon-1/4} \frac{(1-2^{1-s})^2\zeta^2(s)x^s}{s^2}\,\mathrm{d}s \ll x^{\varepsilon-1/4},$$

hence (13) follows from (14), (15) and the residue theorem.

In concluding, note that if we write

$$\varphi(x) = \tfrac{1}{4}\log x + A + \varphi_1(x),$$

where A is given by (15) then, for $\Re\mathrm{e}\, s = \sigma > 0$, (4) yields

$$s^2\left(\frac{A}{s} + \frac{1}{4s^2} + \int_1^{\infty} \varphi_1(x)x^{-s-1}\,\mathrm{d}x\right) = (1-2^{1-s})^2\zeta^2(s),$$

and the above integral converges absolutely, for $\sigma > -1/4$, in view of (13). Thus by analytic continuation it follows that, for $\sigma > -1/4$,

$$As + \tfrac{1}{4} + s^2\int_1^{\infty} \varphi_1(x)x^{-s-1}\,\mathrm{d}x = (1-2^{1-s})^2\zeta^2(s).$$

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