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THE FRACTAL INTERPOLATION FOR COUNTABLE SYSTEMS OF DATA

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In this paper we will extend the fractal interpolation from the finite case to the case of countable sets of data. The main result is that, given an countable system of data in $[a, b] \times Y$, where $[a, b]$ is a real interval and Y a compact and arcwise connected metric space, there exists a countable iterated function system whose attractor is the graph of a fractal interpolation function.

1. PRELIMINARIES

We shall present some notions and results used in the sequel (more complete and rigorous treatments may be found in [**4**], [**2**], [**5**]).

1.1. Hausdorff metric. Let (X, d) be a complete metric space and $\mathcal{K}(X)$ be the class of all compact non-empty subsets of X.

The function $h : \mathcal{K}(X) \times \mathcal{K}(X) \longrightarrow \mathbb{R}_+$,

$$h(A, B) = \max\{\mathrm{d}(A, B), \mathrm{d}(B, A)\},$$

where

$$\mathrm{d}(A, B) = \sup_{x \in A} \big(\inf_{y \in B} \mathrm{d}(x, y) \big), \quad \text{for all } A, B \in \mathcal{K}(X),$$

is called the *Hausdorff metric*.

The set $\mathcal{K}(X)$ is a complete metric space with respect to this metric h.

1.2. Iterated Function Systems. Let (X, d) be a complete metric space.

A set of contractions $(\omega_n)_{n=1}^{N}$, $N \geq 1$, is called an *iterated function system* (IFS), according to M. Barnsley [**2**]. Such a system of maps induces a set function $\mathcal{S}_N : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$,

$$\mathcal{S}_N(E) = \bigcup_{n=1}^{N} \omega_n(E)$$

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which is a contraction on $\mathcal{K}(X)$ with contraction ratio $r \leq \max_{1\leq n\leq N} r_n$, r_n being the contraction ratio of $\omega_n,\ n=1,\ldots,N$.

According to the Banach contraction principle, there is a unique set $A_N \in \mathcal{K}(X)$ which is invariant with respect to $\mathcal{S}_N$, that is

$$A_N = \mathcal{S}_N(A_N) = \bigcup_{n=1}^{N} \omega_n(A_N).$$

We say that the set A_N is the *attractor* of the IFS $(\omega_n)_{n=1}^{k}$.

1.3. Countable Iterated Function Systems. Suppose that (X, d) is a compact metric space.

A sequence of contractions $(\omega_n)_{n\geq 1}$ on X whose contraction ratios are, respectively, $r_n > 0$, such that $\sup_n r_n < 1$ is called a *countable iterated function system*, for simplicity CIFS.

Let $(\omega_n)_{n\geq 1}$ be a CIFS.

We define the set function $\mathcal{S} : \mathcal{K}(X)\mathcal{K}(X)$, by

$$\mathcal{S}(E) = \overline{\bigcup_{n\geq 1} \omega_n(E)},$$

where the bar means the closure of the corresponding set. Then, $\mathcal{S}$ is a contraction map on $(\mathcal{K}(X), h)$ with contraction ratio $r \leq \sup_n r_n$. According to the Banach contraction principle, there exists a unique non-empty compact set $A \subset X$ which is invariant for the family $(\omega_n)_{n\geq 1}$, that is

$$A = \mathcal{S}(A) = \overline{\bigcup_{n\geq 1} \omega_n(A)}.$$

The set A is called *the attractor* of CIFS $(\omega_n)_{n\geq 1}$.

We denote by A_k and, respectively, by $\mathcal{S}_k$ the attractor and the contraction associated to the partial IFS $(\omega_n)_{n=1}^{k}$, for $k \geq 1$.

The attractor of CIFS $(\omega_n)_{n\geq 1}$ is $A = \overline{\bigcup_{k\geq 1} A_k} = \lim_k A_k$, the limit being taken in $(\mathcal{K}(X), h)$.

Hence, the attractor of CIFS $(\omega_n)_{n\geq 1}$ is approximated by the attractors of the partial IFS $(\omega_n)_{n=1}^{k}$, $k \geq 1$.

2. COUNTABLE FRACTAL INTERPOLATION

In this section we will introduce and describe the countable system of data and the corresponding interpolation functions. Then we will construct the CIFS associated to that system of data by generalizing a construction of Barnsley

[1], [2]. We will show that the attractor of that CIFS is the graph of a fractal interpolation function.

Let (Y, d_Y) be a compact and arcwise connected metric space.

Definition 1. *A Countable System of Data* (*abbreviated CSD*) *is a set of points of the form*

$$(x_n, F_n)_{n\geq 0} = \{(x_n, F_n) \in \mathbb{R} \times Y : \ n = 0, 1, \ldots\}$$

where the sequence $(x_n)_{n\geq 0}$ *is strictly increasing and bounded and* $(F_n)_{n\geq 0}$ *is convergent.*

In the sequel $(x_n, F_n)_{n\geq 0}$ will be a CSD and we denote by $a = x_0$, $b = \lim\limits_{n} x_n$, $M = \lim\limits_{n} F_n$ and by $X = [a, b] \times Y$. Clearly $(b, M) \in X$.

Definition 2. *An interpolation function corresponding to this CSD is a continuous function* $f : [a, b] \to Y$ *such that*

$$f(x_n) = F_n \quad \textit{for } n = 0, 1, \ldots.$$

The points $(x_n, F_n) \in [a, b] \times Y$, $n \geq 0$, *are called the interpolation points.*

Remarks: 1° It is clear that $f(b) = M$, since f is continuous.

2° For each countable system of data there exists an interpolation function. For example, we can construct the interpolation function as follows: by a standard topological fact, for each $n \geq 1$, there exists a continuous map $f_n : [x_{n-1}, x_n] \to Y$ such that $f_n(x_{n-1}) = F_{n-1}$ and $f_n(x_n) = F_n$. Then put

$$f(x) = \begin{cases} f_n(x), & \text{for } x \in [x_{n-1}, x_n),\ n = 1, 2, \ldots\ ; \\ M, & \text{for } x = b. \end{cases}$$

Define a metric δ on X by

$$\delta\big((y_1, z_1), (y_2, z_2)\big) = |\, y_1 - y_2\,| + \theta\, \mathrm{d}_Y(z_1, z_2), \tag{1}$$

for all points $(y_1, z_1), (y_2, z_2) \in X$, where θ is a positive real number to be specified below. It is obvious that δ is a metric on X. Also, note that the metric space (X, δ) is compact.

Let c and s be real numbers, with $0 \leq s < 1$ and $c > 0$. For each $n = 1, 2, \ldots$, let $\varphi_n : X \to Y$ be a function which satisfies the inequalities

$$\mathrm{d}_Y\big(\varphi_n(\alpha, y), \varphi_n(\beta, y)\big) \leq c\, |\alpha - \beta| \quad \text{for any } \alpha, \beta \in [a, b] \text{ and } y \in Y, \tag{2}$$

and

$$\mathrm{d}_Y\big(\varphi_n(\alpha, y), \varphi_n(\alpha, z)\big) \leq s \mathrm{d}_Y(y, z) \quad \text{for any } \alpha \in [a, b] \text{ and } y, z \in Y. \tag{3}$$

Define a transformation $\omega_n : X \to X$ by

$$\omega_n(x, y) = \big(l_n(x), \varphi_n(x, y)\big), \ \text{ for all } (x, y) \in X,\ n = 1, 2, \ldots,$$

where $l_n : [a, b] \to [x_{n-1}, x_n]$ is the invertible transformation $l_n(x) = a_n x + e_n$,

$$a_n = \frac{x_n - x_{n-1}}{b - a} \text{ and } e_n = \frac{bx_{n-1} - ax_n}{b - a}. \tag{4}$$

Notice that

$$a_n > 0 \text{ for all } n \geq 1 \text{ and } \sup_{n\geq 1} a_n < 1. \tag{5}$$

Theorem 1. *Let the family* $(\omega_n)_{n\geq 1}$ *be defined as above. Assume that there are real constants* c *and* s *such that* $c > 0$, $0 \leq s < 1$ *and conditions* (2) *and* (3) *are fullfield. Let the constant* θ *in the definition of the metric* δ *in equation* (1) *be defined by*

$$\theta = \frac{\inf_{n\geq 1}(1 - a_n)}{2c}.$$

Then $(\omega_n)_{n\geq 1}$ *is a CIFS with respect to the metric* δ. *In particular, there exists a unique nonempty compact set* $A \subset X$ *such that*

$$A = \overline{\bigcup_{n\geq 1} \omega_n(A)}.$$

Proof. We will show that the transformations ω_n, $n \geq 1$, are contraction maps on (X, δ) having the supremum of contraction ratios less than 1.

Let $n \geq 1$ and $(y_1, z_1), (y_2, z_2) \in X$. Using (2), (3), (5) we have

$$\begin{aligned}
&\delta\big(\omega_n(y_1, z_1), \omega_n(y_2, z_2)\big) \\
&\quad = \delta\Big(\big(a_n y_1 + e_n, \varphi_n(y_1, z_1)\big)\,,\,\big(a_n y_2 + e_n, \varphi_n(y_2, z_2)\big)\Big) \\
&\quad = a_n\,|y_1 - y_2|\, + \theta \mathrm{d}_Y(\varphi_n(y_1, z_1), \varphi_n(y_2, z_2)) \\
&\quad \leq (a_n + \theta c)\,|y_1 - y_2|\, + \theta s \mathrm{d}_Y(z_1, z_2) \leq \max\{\alpha, s\}\delta\big((y_1, z_1), (y_2, z_2)\big),
\end{aligned}$$

where $\alpha = \sup_{n\geq 1} a_n + \theta c < 1$.

Thus $(\omega_n)_{n\geq 1}$ is a CIFS on (X, δ). Then there exists a nonempty compact subset of (X, δ) which is the attractor of $(\omega_n)_{n\geq 1}$.

□

We now constrain the CIFS $(\omega_n)_{n\geq 1}$ on X, defined above, to ensure that the attractor includes the CSD. We assume that

$$\varphi_n(x_0, F_0) = F_{n-1} \text{ and } \varphi_n(b, M) = F_n \text{ for } n = 1, 2, \ldots. \tag{6}$$

Then it follows that

$$\omega_n(x_0, F_0) = (x_{n-1}, F_{n-1}) \text{ and } \omega_n(b, M) = (x_n, F_n) \text{ for } n = 1, 2, \ldots. \tag{7}$$

Definition 3. *We say that the CIFS $(\omega_n)_{n\geq 1}$ defined above is associated with the CSD $(x_n, F_n)_{n\geq 0}$.*

Theorem 2. *Let $(\omega_n)_{n\geq 1}$ denote the CIFS associated with the CSD $(x_n, F_n)_{n\geq 0}$. In particular, assume that there are real constants c and s such that $0 \leq s < 1$, $c > 0$, and conditions (2), (3) and (6) are fullfied. Then there exists an interpolation function f corresponding to the CSD such that the graph of f is the attractor of CIFS $(\omega_n)_{n\geq 1}$. That is*

$$A = \{\big(x, f(x)\big) : \ x \in [a, b]\}.$$

Proof. Let $\mathcal{F}$ denote the set of continuous function $f : [a, b] \to Y$ such that $f(a) = F_0$ and $f(b) = M$. The set $\mathcal{F}$ is non empty (see the above remark).

We consider the uniform metric d on $\mathcal{F}$,

$$\mathrm{d}(f, g) = \sup_{x\in[a,b]} \mathrm{d}_Y\big(f(x), g(x)\big), \quad \text{for all } f, g \in \mathcal{F}.$$

It is a standard fact that $(\mathcal{F}, \mathrm{d})$ is a complete metric space.

Define a mapping $T : \mathcal{F} \to \mathcal{F}$ by

$$(Tf)(x) = \varphi_n\Big(l_n^{-1}(x),\ f\big(l_n^{-1}(x)\big)\Big) \text{ for } x \in [x_{n-1}, x_n),\ n = 1, 2, \ldots,$$

and

$$(Tf)(b) = M$$

where, for each $n = 1, 2, \ldots$, $l_n : [a, b] \to [x_{n-1}, x_n]$ is the invertible transformation $l_n(x) = a_n x + e_n$ defined as above.

Notice that, from (4)

$$l_n^{-1}(x_{n-1}) = x_0, \quad l_n^{-1}(x_n) = b, \quad \forall\, n = 1, 2, \ldots. \tag{8}$$

To verify that $T(\mathcal{F}) \subset \mathcal{F}$, let $f \in \mathcal{F}$. Then, using (6), (8),

$$\begin{aligned}(Tf)(a) = (Tf)(x_0) \ &= \ \varphi_1\Big(l_1^{-1}(x_0), f\big(l_1^{-1}(x_0)\big)\Big)\\ &= \ \varphi_1\big(x_0, f(x_0)\big) = \varphi_1(x_0, F_0) = F_0\end{aligned}$$

and, by definition,

$$(Tf)(b) = M.$$

The continuity of application Tf on the intervals $[x_{n-1}, x_n)$, $n = 1, 2, \ldots$, is obvious (because the applications φ_n, $n \geq 1$, are continuous, by (2) and (3). Then it remains to prove that Tf is continuous at each of the points $x_1, x_2, \ldots$ and that Tf is left continuous in b.

Let $n \geq 1$. We have

$$\lim_{x\searrow x_n} (Tf)(x) \ = \ \varphi_{n+1}\Big(l_{n+1}^{-1}(x_n), f\big(l_{n+1}^{-1}(x_n)\big)\Big)$$

$$
\begin{aligned}
&= \varphi_{n+1}\big((x_0), f(x_0)\big) = F_n, \\
\lim_{x \nearrow x_n} (Tf)(x) &= \varphi_n\Big(l_n^{-1}(x_n), f\big(l_n^{-1}(x_n)\big)\Big) \\
&= \varphi_n\big(b, f(b)\big) = F_n = (Tf)(x_n).
\end{aligned}
$$

Next, by using the continuity of Tf on $[a,b)$, the fact that $x_n \nearrow b$ and the definition of Tf, we deduce that $\lim\limits_{x \nearrow b}(Tf)(x) = M$. We conclude that T does indeed map $\mathcal{F}$ into $\mathcal{F}$.

Now, we will show that T is a contraction mapping on the metric space $(\mathcal{F}, \mathrm{d})$. Let $f, g \in \mathcal{F}$. Let $x \in [a,b)$ and $n \geq 1$ such that $x \in [x_{n-1}, x_n]$.

Then

$$
\begin{aligned}
\mathrm{d}_Y\big((Tf)(x), (Tg)(x)\big) &= \mathrm{d}_Y\bigg(\varphi_n\Big(l_n^{-1}(x), f\big(l_n^{-1}(x)\big)\Big), \varphi_n\Big(l_n^{-1}(x), g\big(l_n^{-1}(x)\big)\Big)\bigg) \\
&\leq s\,\mathrm{d}_Y\Big(f\big(l_n^{-1}(x)\big), g\big(l_n^{-1}(x)\big)\Big) \leq s\,\mathrm{d}(f,g).
\end{aligned}
$$

Since the above relations are clearly verified for $x = b$, it follows that

$$\mathrm{d}(Tf, Tg) \leq s\,\mathrm{d}(f,g).$$

The Contraction Mapping Theorem implies that T possesses a unique fixed point in $\mathcal{F}$. That is there exists a function $f \in \mathcal{F}$ such that $(Tf)(x) = f(x)$, for all $x \in [a,b]$.

Taking into account the definition of T and the relations (8), it is easy to verify that if $f_0 \in \mathcal{F}$, then Tf_0 is an interpolation function for CSD $(x_n, F_n)_{n\geq 0}$. Particularly f is an interpolation function for CSD $(x_n, F_n)_{n\geq 0}$.

Let G denote the graph of f. We will prove the equality

$$G = \overline{\bigcup_{n\geq 1} \omega_n(G)}$$

by proving both inclusions.

Let $x \in [a,b]$ and $n \geq 1$. Then, by (4), $a_n x + e_n \in [x_{n-1}, x_n]$, hence

$$
\begin{aligned}
\omega_n\big(x, f(x)\big) &= \big(a_n x + e_n, \varphi_n(x, f(x))\big) = \big(a_n x + e_n, (Tf)(a_n x + e_n)\big) \\
&= \big(a_n x + e_n, f(a_n x + e_n)\big) \in G.
\end{aligned}
$$

Because G is closed set and $\omega_n(G) \subset G$ one obtains: $\overline{\bigcup\limits_{n\geq 1} \omega_n(G)} \subset G$.

Conversely, let $\big(x, f(x)\big) \in G$. Suppose that $x \in [a,b)$ and let $n \geq 1$ such that $x \in [x_{n-1}, x_n]$. Then

$$\big(x, f(x)\big) = \omega_n\Big(l_n^{-1}(x), f\big(l_n^{-1}(x)\big)\Big) \in \omega_n(G).$$

Next, since $(x_n, f(x_n)) \in \omega_n(G)$ for all $n \geq 1$, it follows that

$$\big(b, f(b)\big) = \lim_n \big(x_n, f(x_n)\big) \in \overline{\bigcup_{n\geq 1} \omega_n(G)}.$$

By uniqueness of the attractor of an CIFS, we deduce that $A = G$. This completes the proof.

$\square$

Definition 4. *The application f defined in the above theorem is called the* fractal interpolation function *associated with the CSD $(x_n, F_n)_{n\geq 0}$.*

The following theorems give two approximation methods for the graph of the fractal interpolation function given in Theorem 2.

Theorem 3. *In the conditions of Theorem 2, using the same notations, we define as usually $\mathcal{S} : \mathcal{K}(X) \longrightarrow \mathcal{K}(X)$, by $\mathcal{S}(E) = \overline{\bigcup_{n\geq 1} \omega_n(E)}$.*

If f_0 is a map in $\mathcal{F}$ whose graph is A_0, then the graph of Tf_0 is $\mathcal{S}(A_0)$.

Moreover, if A_p denote the graph of the interpolation function $T^p f_0$ ($p = 1, 2, \ldots$), then the sequence $(A_p)_p$ converges with respect to the Hausdorff metric to the attractor of CIFS $(\omega_n)_{n\geq 1}$, hence to the graph of a fractal interpolation function (we have denoted $T^1 = T$, $T^{p+1} = T \circ T^p$, $p \geq 1$).

Proof. We will prove the equality

$$\mathcal{S}(A_0) = \overline{\bigcup_{n\geq 1} \omega_n(A_0)} = \{(t, Tf_0(t)) :\ t \in [a, b]\} \tag{9}$$

by proving both inclusions.

Let $y \in \mathcal{S}(A_0)$. First assume that $y \in \bigcup_{n\geq 1} \omega_n(A_0)$. Then there exists $n \geq 1$ such that $y \in \omega_n(A_0)$, hence there exists $x \in [a, b]$ such that

$$y = \omega_n\big(x, f_0(x)\big) = \Big(a_n x + e_n, \varphi_n\big(x, f_0(x)\big)\Big).$$

By using (4), we have

$$x_{n-1} = a_n x_0 + e_n \leq a_n x + e_n \leq a_n b + e_n = x_n. \tag{10}$$

Next, for $t = a_n x + e_n$, one has $t \in [a, b]$, $y = (t, Tf_0(t))$.

Now, if

$$y \in \overline{\bigcup_{n\geq 1} \omega_n(A_0)} \text{ and } y_k \in \bigcup_{n\geq 1} \omega_n(A_0),\ \ y_k \xrightarrow{k} y,$$

there is $t_k \in [a, b]$, such that $y_k = (t_k, Tf_0(t_k))$, for all $k = 1, 2, \ldots$. It follows that $y = \lim_k (t_k, Tf_0(t_k))$ belongs to the graph of Tf_0.

Conversely, for $y = \big(t, Tf_0(t)\big)$, $t \in [a, b)$, there is $n \geq 1$ such that $x_{n-1} \leq t \leq x_n$. Then (see (10)) there exists $x \in [a, b]$ so that $t = a_n x + e_n$ and hence

$$y = \omega_n\big(x, f_0(x)\big) \in \mathcal{S}(A_0).$$

Also $y = \big(b, Tf_0(b)\big) = \lim\limits_{x \nearrow b}(x, Tf_0(x)) \in \mathcal{S}(A_0)$.

Now, iterating (9), it follows $\mathcal{S}^p(A_0) = A_p$. Since $(\mathcal{S}^p(A_0))_p$ converges with respect to the Hausdorff metric to the attractor of CIFS, by Theorem 2 it follows the last assertion.

Theorem 4. *In the above conditions, the graph of the fractal interpolation function corresponding to the CIFS $(\omega_n)_{n\geq 1}$ associated with CSD is approximated with respect to the Hausdorff metric by the attractors of the partial IFS $(\omega_n)_{n=1}^k$, $k \geq 1$.*

Proof. The assertion follows immediately from Theorem 2 and the section 1.3 (see [**5**, Cor. 2.2]).

Particular case: the countable system of data in $\mathbb{R}^2$

We consider now the case when Y is a compact nonempty subset of $\mathbb{R}$. Thus a CSD on $X = [a, b] \times Y$ is a set of points of the form

$$(x_n, F_n)_{n\geq 0} = \{(x_n, F_n) \in \mathbb{R}^2 : \ n = 0, 1, 2, \ldots\},$$

where $a = x_0 < x_1 < \ldots$, $b = \lim\limits_n x_n$, $(F_n)_{n\geq 0}$ is a convergent sequence in Y and $M = \lim\limits_n F_n$.

The maps φ_n, $n \geq 1$, in the construction of $(\omega_n)_{n\geq 1}$ can be the affine transformations:

$$\varphi_n(x, y) = c_n x + d_n y + g_n, \ \ c_n, d_n, g_n \in \mathbb{R}, \ \ n = 1, 2, \ldots.$$

The relations (7) will be

$$\begin{aligned} a_n x_0 + e_n &= x_{n-1}, \\ a_n b + e_n &= x_n, \\ c_n x_0 + d_n F_0 + g_n &= F_{n-1}, \\ c_n b + d_n M + g_n &= F_n. \end{aligned} \tag{11}$$

It follows that there is effectively one free parameter in each transformation. For the sequel we choose this parameter to be d_n (d_n is called the *vertical scaling factor* in the transformation ω_n).

Thus, if d_n is any real number, we obtain from (11), using the above notations,

$$\begin{aligned} a_n &= \frac{x_n - x_{n-1}}{b - a}, \\ e_n &= \frac{b x_{n-1} - x_0 x_n}{b - a}, \\ c_n &= \frac{F_n - F_{n-1}}{b - a} - d_n \frac{M - F_0}{b - a}, \\ g_n &= \frac{b F_{n-1} - x_0 F_n}{b - a} - d_n \frac{b F_0 - x_0 M}{b - a}. \end{aligned} \tag{12}$$

It follows that, if we assume that the vertical scaling factor d_n satisfies the conditions $d_n \geq 0$, for $n = 1, 2, \ldots$, and $\sup_{n \geq 0} d_n < 1$, then there is a metric δ on X, equivalent to the Euclidean metric d, such that $(\omega_n)_{n \geq 1}$ is a CIFS on the compact metric space (X, δ).

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