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ON A SUM INVOLVING THE PRIME COUNTING FUNCTION $\pi(x)$

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An asymptotic formula for the sum of reciprocals of $\pi(n)$ is derived, where $\pi(x)$ is the number of primes not exceeding x. This result improves the previous results of De Koninck-Ivić and L. Panaitopol.

Let, as usual, $\pi(x) = \sum_{p\le x} 1$ denote the number of primes not exceeding x. The prime number theorem (see e.g., [**2**, Chapter 12]) in its strongest known form states that

$$\pi(x) = \operatorname{li} x + R(x), \tag{1}$$

with

$$\operatorname{li} x := \int_2^x \frac{\mathrm{d}t}{\log t} = x\left(\frac{1}{\log x} + \frac{1!}{\log^2 x} + \cdots + \frac{m!}{\log^{m+1} x} + O\Big(\frac{1}{\log^{m+2} x}\Big)\right) \tag{2}$$

for any fixed integer $m \ge 0$, and

$$R(x) \ll x \exp\big(-C\delta(x)\big), \quad \delta(x) := (\log x)^{3/5}(\log\log x)^{-1/5} \quad (C > 0), \tag{3}$$

where henceforth $C, C_1, \dots$ will denote absolute constants. In [**1**, Theorem 9.1] J.-M. De Koninck and the author proved that

$$\sum_{2\le n\le x} \frac{1}{\pi(n)} = \frac{1}{2}\log^2 x + O(\log x). \tag{4}$$

Recently L. Panaitopol [**1**] improved (4) to

$$\sum_{2\le n\le x} \frac{1}{\pi(n)} = \frac{1}{2}\log^2 x - \log x - \log\log x + O(1). \tag{5}$$

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One obtains (5) from the asymptotic formula

$$\frac{1}{\operatorname{li} x} = \frac{1}{x}\left(\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \cdots - \frac{k_m\bigl(1+\alpha_m(x)\bigr)}{\log^m x}\right), \tag{6}$$

where $\alpha_m(x) \ll_m 1/\log x$, and the constants $k_1, \ldots, k_m$ are defined by the recurrence relation

$$k_m + 1!\,k_{m-1} + \cdots + (m-1)!\,k_1 = m\cdot m! \qquad (m \in \mathbf{N}), \tag{7}$$

so that $k_1 = 1$, $k_2 = 3$, $k_3 = 13$, etc. This was established in [**3**]. Using (6) we shall give a further improvement of (4), contained in the following

Theorem. *For any fixed integer $m \ge 2$ we have*

$$\begin{aligned}\sum_{2\le n\le x} \frac{1}{\pi(n)} &= \frac{1}{2}\log^2 x - \log x - \log\log x + C\\ &\quad + \frac{k_2}{\log x} + \frac{k_3}{2\log^2 x} + \cdots + \frac{k_m}{(m-1)\log^{m-1} x} + O\left(\frac{1}{\log^m x}\right),\end{aligned} \tag{8}$$

where C is an absolute constant, and $k_2, \ldots, k_m$ are the constants defined by (7).

Proof. From (1) we have

$$\begin{aligned}\sum_{2\le n\le x} \frac{1}{\pi(n)} &= 1 + \sum_{3\le n\le x}\frac{1}{\operatorname{li} n} - \sum_{3\le n\le x}\frac{R(n)}{\operatorname{li} n\,\bigl(\operatorname{li} n + R(n)\bigr)}\\ &= \sum_{3\le n\le x}\frac{1}{\operatorname{li} n} + \left(1 - \sum_{n=3}^{\infty}\frac{R(n)}{\operatorname{li} n\,\bigl(\operatorname{li} n + R(n)\bigr)}\right) + \sum_{n>x}\frac{R(n)}{\operatorname{li} n\,(\operatorname{li} n + R(n))}\\ &= \sum\nolimits_1 + C_1 + \sum\nolimits_2,\end{aligned}$$

say. By using the bound $\operatorname{li} x \ll x/\log x$ and (3) it is seen that

$$\begin{aligned}\sum\nolimits_2 &= \sum_{n>x}\frac{R(n)}{\operatorname{li} n\,\bigl(\operatorname{li} n + R(n)\bigr)} \ll \sum_{n>x}\frac{1}{n}\,e^{-C\delta(n)/2}\\ &\ll e^{-C\delta(x)/3}\int_{x-1}^{\infty}\frac{1}{t}\,e^{-C\delta(t)/6}\mathrm{d}t \ll e^{-C\delta(x)/3},\end{aligned}$$

since $\delta(x)$ is increasing for $x \ge x_0$, and the substitution $\log t = u$ easily shows that the above integral is convergent. To evaluate $\sum_1$ we need the familiar Euler–Maclaurin summation formula (see e.g., [**2**, eq. (A.23)]) in the form

$$\sum_{X<n\le X} f(n) = \int_X^Y f(t)\,\mathrm{d}t - \psi(Y)f(Y) + \psi(X)f(X) + \int_X^Y \psi(t)f'(t)\,\mathrm{d}t, \tag{9}$$

where $\psi(x) = x - [x] - 1/2$ and $f(x) \in C^1[X, Y]$. We obtain from (6), for $m \ge 2$ a fixed integer,

$$
\begin{aligned}
\textstyle\sum_1 &= \sum_{3\le n\le x} \frac{1}{\operatorname{li} n} \\
&= \sum_{3\le n\le x} \frac{1}{n}\left(\log n - 1 - \frac{k_1}{\log n} - \frac{k_2}{\log^2 n} - \cdots - \frac{k_m\bigl(1+\alpha_m(n)\bigr)}{\log^m n}\right),
\end{aligned} \tag{10}
$$

and we evaluate each sum in (10) by using (9). We obtain

$$
\begin{aligned}
\sum_{3\le n\le x} \frac{\log n}{n} &= \frac{1}{2}\log^2 x + c_1 + O\Bigl(\frac{\log x}{x}\Bigr),\\
\sum_{3\le n\le x} \frac{1}{n} &= \log x + c_2 + O\Bigl(\frac{1}{x}\Bigr),\\
\sum_{3\le n\le x} \frac{k_1}{n\log n} &= \log\log x + c_3 + O\Bigl(\frac{1}{x\log x}\Bigr),
\end{aligned}
$$

and for $2 \le r \le m$

$$
\begin{aligned}
\sum_{3\le n\le x} \frac{k_r}{n\log^r n} &= k_r\int_3^x \frac{\mathrm{d}t}{t\log^r t} + C_r + O\Bigl(\frac{1}{x\log^r x}\Bigr)\\
&= k_r\int_3^\infty \frac{\mathrm{d}t}{t\log^r t} - k_r\int_x^\infty \frac{\mathrm{d}t}{t\log^r t} + C_r + O\Bigl(\frac{1}{x\log^r x}\Bigr)\\
&= D_r - \frac{k_r}{(r-1)\log^{r-1} x} + O\Bigl(\frac{1}{x\log^r x}\Bigr)
\end{aligned}
$$

with

$$
D_r = C_r + k_r\int_3^\infty \frac{\mathrm{d}t}{t\log^r t}\,.
$$

Finally in view of $\alpha_m(x) \ll 1/\log x$ it follows that, for $m \ge 2$ fixed,

$$
\sum_{3\le n\le x} \frac{k_m\alpha_m(n)}{n\log^m n} = \sum_{n=3}^\infty \frac{k_m\alpha_m(n)}{n\log^m n} + O\Bigl(\frac{1}{\log^m x}\Bigr).
$$

Putting together the above expressions in (10) we infer that

$$
\begin{aligned}
\textstyle\sum_1 &= \frac{1}{2}\log^2 x - \log x - \log\log x + C\\
&\quad + \frac{k_2}{\log x} + \frac{k_3}{2\log^2 x} + \cdots + \frac{k_m}{(m-1)\log^{m-1} x} + O\Bigl(\frac{1}{\log^m x}\Bigr),
\end{aligned}
$$

and then (8) easily follows with

$$
C = C_1 + c_1 - c_2 - c_3 - D_2 - \cdots - D_m - \sum_{n=3}^\infty \frac{k_m\alpha_m(n)}{n\log^m n}\,.
$$

The constant C in (8) does not depend on m, which can be easily seen by taking two different values of m and then comparing the results.

Note that we can evaluate directly $\sum_1$ by the EULER-MACLAURIN summation formula to obtain

$$\sum\nolimits_1 = \int_3^x \frac{\mathrm{d}t}{\mathrm{li}\, t} + C_0 + O\Big(\frac{\log x}{x}\Big). \tag{11}$$

Integration by parts gives, for $x > 3$,

$$\int_3^x \frac{\mathrm{d}t}{\mathrm{li}\, t} = \int_3^x \log t \, \mathrm{d}(\log\ \mathrm{li}\, t) = \log x \log(\mathrm{li}\, x) - \int_3^x \frac{\log(\mathrm{li}\, t)}{t}\, \mathrm{d}t - \log 3 \log \mathrm{li}\, 3,$$

which inserted into (11) gives another expression for our sum, namely

$$\sum_{2\le n\le x} \frac{1}{\pi(n)} = \log x \log(\mathrm{li}\, x) - \int_3^x \frac{\log(\mathrm{li}\, t)}{t}\, \mathrm{d}t + B + O\big(e^{-D\delta(x)}\big)\ (D > 0), \tag{12}$$

from which we can again deduce (8) by using (2). The advantage of (12) is that it has a sharper error term than (8), but on the other hand the expressions on the right-hand side of (12) involve the non-elementary function $\mathrm{li}\, x$. Note also that the RIEMANN hypothesis (that all complex zeros of the RIEMANN zeta-function $\zeta(s)$ have real parts equal to $\frac{1}{2}$) is equivalent to the statement (see [**2**]) that, for any given $\varepsilon > 0$, $R(x) \ll x^{1/2+\varepsilon}$ in (3), which would correspondingly improve the error term in (12) to $O(x^{-1/2+\varepsilon})$.

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