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ON EQUIVALENCE AND SPECTRAL MULTIPLICITY OF SOME GAUSSIAN PROCESSES

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In this paper we consider some Gaussian second-order stochastic processes (continuous left and purely nondeterministic), in a separable HILBERT space and analyze conditions for these processes to be equivalent. Also, we connect some results of H. CRAMER (from *Structural and statistical problems for a class of stochastic processes*, Princeton Univ. Press, Princeton, NJ, 1971) concerning the problem of spectral multiplicity.

1. INTRODUCTION

Let $x(t)$, $t \in (a,b) \subset \mathbf{R}$ be a second-order real-valued process with $Ex(t) = 0$ for each t. Let $H(x,t)$ be the linear closure generated by $x(s)$, $s \in (a,t]$ in the HILBERT space H of all random variables with finite variance ($Ex^2(t) < \infty$). We will suppose that $x(t)$, $t \in (a,b)$ is continuous left and purely nondeterministic (i.e. $\cap_{t>a} H(x,t) = 0$). It is well known (see [**1**]) that there is a representation:

$$x(t) = \sum_{n=1}^{N} \int_a^t g_n(t,u)\,\mathrm{d}z_n(u), \quad u \le t, \ \ t \in (a,b), \tag{1}$$

where:

1. The processes $z_n(u)$, $n = 1,\dots,N$ are mutually orthogonal with orthogonal increments such that $Ez_n(u) = 0$ and $Ez_n^2(u) = F_n(u)$, where $F_n(u)$, $n = 1,\dots,N$ are non decreasing functions left continuous everywhere on (a,b).
2. The non-random functions $g_n(t,u)$, $u \le t$, are such that:

$$Ex^2(t) = \sum_{n=1}^{N} \int_a^t g_n^2(t,u)\,\mathrm{d}F_n(u) < \infty, \quad \text{for each } t \in (a,b),$$

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3. $\mathrm{d}F_1 > \mathrm{d}F_2 > \cdots > \mathrm{d}F_n$, where the relation $>$ means absolute continuity between measures.

4. $H(x,t) = \sum_{n=1}^{N} \oplus H(z_n,t), \quad t \in (a,b)$.

The expansion (1) satisfying the conditions 1, 2, 3 and 4 is the *canonical representation* for the process $x(t)$. The number N (finite or infinite) is called the *multiplicity* of $x(t)$, and N is uniquely determined by the process $x(t)$. But, the processes $z_n(u)$ and the functions $g_n(t,u)$ are not uniquely determined.

Let $x(t)$ be a Gaussian process given by one integral representation:

$$x(t) = \int_a^t g(t,u)\,\mathrm{d}z(u), \quad u \le t, \ \ t \in [a,b], \tag{2}$$

where the kernel $g(t,u)$ and Gaussian process $z(u)$ satisfy the conditions 1 and 2. This representation may not be canonical. The main question here is to determine spectral multiplicity of $x(t)$. Before we consider this problem let us denote some very well known facts about Gaussian processes.

If we take $x(t) = x(w,t)$, $w \in \Omega$, $t \in [a,b] = T \subset \mathbf{R}$, as a measurable mapping of the basic probability space (Ω, U, P) into the measurable space (X, β, P_x) which to each $w \in \Omega$, corresponds the trajectory $x(w,t) \in X$, $t \in T$, we may now consider the probability space (X, β, P_x) instead of the space (Ω, U, P), where the probability measure is:

$$P_x(B) = P\big(w : x(w,t) \in B, \ B \in \beta\big),$$

β is a Borel σ-field spanned by the cylindric sets $\{x(t) : [x(t_1), \dots, x(t_n)] \in \mathbf{C}\}$, and C is a Borel set from $\mathbf{R}^n$. When a stochastic process $x(t)$, $t \in T$, is a Gaussian (all its finite distributions are Gaussian), then the probability P_x is called a Gaussian measure.

If P_{x_1} and P_{x_2} are two Gaussian measures on the space (X, β), it is well known, they are either equivalent (mutually absolutely continuous) or orthogonal $\big(\exists A \in \beta : P_{x_1}(A) = 1$ and $P_{x_2}(A) = 0\big)$. In the case of equivalence of two Gaussian measures P_{x_1} and P_{x_2} induced from $x_1(t)$ and $x_2(t)$ we say that these Gaussian processes $x_1(t)$ and $x_2(t)$ are equivalent and converse.

According to the fact that a Gaussian process is uniquely determined by the mean $Ex(t)$, $t \in T$, and the covariance function $B(s,t) = E\big(x(s) - Ex(s)\big)\big(x(t) - Ex(t)\big)$, s, $t \in T$, in order to find conditions for equivalence of two Gaussian processes, it is sufficient to consider two particular cases: *a*) the case of different means but the same covariance functions; and *b*) the case of the same means and different covariance functions (see [**5**]). Here it will be considered the case *b*) because we have assumed that for our processes $Ex(t) = 0$ for each t.

In this case (see [**5**]) two Gaussian processes $x_1(t)$ and $x_2(t)$ given by (2), are equivalent if and only if there exists

$$y \in H(z_1) \otimes H(z_2), \ \ y = \int_a^b \int_a^b h(u,v)\,\mathrm{d}z_1(u)\,\mathrm{d}z_2(v),$$

such that

$$\int_a^b \int_a^b h^2(u,v)\,\mathrm{d}F_1(u)\,\mathrm{d}F_2(v) < \infty,$$

and the next equation is satisfied

$$B_1(s,t) - B_2(s,t) = \int_a^b \int_a^b h(u,v)\, g_1(s,u)\, g_2(t,v)\,\mathrm{d}F_1(u)\,\mathrm{d}F_2(v), \quad s,t \in T, \tag{3}$$

where $B_i(s,t)$ are covariance functions of $x_i(t) = \int_a^t g_i(t,u)\,\mathrm{d}z_i(u),\ t \in T$.

For equivalent processes $x_1(t)$ and $x_2(t)$, the spectral multiplicity is the same (see [**5**]). The converse doesn't hold. This fact is shown in the next simple example.

Example. Let $x_1(t)$ be a Wiener process, $x_1(t) = \int_0^t \mathrm{d}z(u),\ u \le t,\ t \in [0,\tau]$, and $x_2(t)$ a Markov process given by

$$x_2(t) = g(t)x_1(t) = g(t)\int_0^t \mathrm{d}z(u),\ u \le t,\ u,t \in [0,\tau],$$

where $g(t) > 0$, $t \in [0,\tau]$, is not absolutely continuous. Then the difference of their covariance functions

$$B_1(s,t) - B_2(s,t) = (1 - g(s)\,g(t))\min(s,t),$$

is not absolutely continuous and we cannot represent this difference in the form (3). It means $x_1(t)$ and $x_2(t)$ are not equivalent. But the spectral multiplicity for $x_1(t)$ and $x_2(t)$ is the same (see [**2**]).

Lemma. *Let us suppose for the process $x(t)$ given by (2), the functions $g(t,u)$ and $\partial g(t,u)/\partial t$ are bounded and continuous for u, $t \in [a,b]$, $u \le t$, and the function $F(u) = Ez^2(u)$ is absolutely continuous with $f(u) = \partial F(u)/\partial u$. Then the covariance function $B(s,t)$ of this process has continuous partial derivatives $\partial B(s,t)/\partial t$ and $\partial B(s,t)/\partial s$ for all s, t except for $s = t$. At $s = t$ there is a jump equal to $g^2(t,t)\,f(t)$.*

Proof. It is well known that the covariance function of such process is

$$B(s,t) = \int_a^{s\wedge t} g(s,u)\,g(t,u)\,f(u)\,\mathrm{d}u.$$

According to the assumption about $g(t,u)$ and $\partial g(t,u)/\partial t$ it is easy to see that $\partial B(s,t)/\partial t$ and $\partial B(s,t)/\partial s$ are continuous partial derivatives for all $s \ne t$. At the diagonal $s = t$, we have to consider two cases. When $\min(s,t) = s$ we have

$$\begin{aligned}\lim_{s\to t} &\frac{B(s,t) - B(t,t)}{s-t} \\ &= \lim_{s\to t}\int_a^s g(t,u)\,\frac{g(s,u) - g(t,u)}{s-t} f(u)\,\mathrm{d}u - \lim_{s\to t}\int_s^t \frac{g^2(t,u)}{s-t} f(u)\,\mathrm{d}u \\ &= \int_a^t g(t,u)\,f(u)\,\partial g(t,u)/\partial t\,\mathrm{d}u + g^2(t,t)\,f(t).\end{aligned}$$

If $\min(s,t)=t$ we obtain

$$\begin{aligned}\lim_{s\to t}\frac{B(s,t)-B(t,t)}{s-t} &= \lim_{s\to t}\int_a^t g(t,u)\,\frac{g(s,u)-g(t,u)}{s-t}f(u)\,\mathrm{d}u\\ &= \int_a^t g(t,u)\,f(u)\,\partial g(t,u)/\partial t\,\mathrm{d}u.\end{aligned}$$

So, there is a jump of the height $g^2(t,t)\,f(t)$ at the diagonal $s=t$ for partial derivatives of $B(s,t)$.

Corollary. *For equivalence of two Gaussian processes $x_1(t)$ and $x_2(t)$, the necessary condition is that the discontinuities of the partial derivatives of $B_1(s,t)$ and $B_2(s,t)$ at the diagonal $s=t$ must be the same* :

$$f_1(t)\,{g_1}^2(t,t)=f_2(t)\,{g_2}^2(t,t). \tag{4}$$

2. MAIN RESULT

One of the problems here is to find out a criteria for processes given by (2) to be multiplicity $N=1$. CRAMER stated in Theorem 5.1. in [**1**], that the *regularity conditions* ensure a multiplicity of unity for a process which has a canonical expansion. Here the main idea is to fortify equivalence of two Gaussian processes from which one has already multiplicity one.

Theorem 1. *Let $x(t)$, $t\in[0,\tau]=T$, be a process given by* (2) *where $z(u)$, $u\in[0,\tau]$, is a* WIENER *process. If $g(t,t)\neq 0$, for all $t\in T$, and $\left(\dfrac{g(t,u)}{g(t,t)}\right)_t'\in L^2(\mathrm{d}t\times\mathrm{d}u)$, i.e.*

$$\int_0^\tau\int_0^\tau\left(\left(\frac{g(t,u)}{g(t,t)}\right)_t'\right)^2\mathrm{d}t\,\mathrm{d}u<\infty, \tag{5}$$

then the process $x(t)$ has multiplicity one.

Proof. Let us introduce the process $y(t)=\int_0^t g(t,t)\,\mathrm{d}z(u)$, $u\le t$, $u,t\in[0,\tau]=T$, where $z(u)$ is a WIENER process. Now, one of the necessary condition for equivalence of $x(t)$ and $y(t)$ is satisfied (see the previous corollary (4)).

The difference between their covariance functions is

$$B_1(s,t)-B_2(s,t)=\int_0^{s\wedge t}\big(g(s,u)\,g(t,u)-g(s,s)\,g(t,t)\big)\,\mathrm{d}u.$$

According to (3) to find out the necessary and sufficient condition for equivalence of $x(t)$ and $y(t)$ we have to solve the next integral equation, regarding $h(u,v)$ as the unknown function:

$$\int_0^{s\wedge t}\left(g(s,u)\frac{g(t,u)}{g(t,t)}-g(s,s)\right)\mathrm{d}u=\int_0^s\int_0^t h(u,v)\,g(s,u)\,\mathrm{d}u\,\mathrm{d}v,\quad s,t\in T.$$

If we suppose $\min(s,t) = s$, after some calculation we obtain for $u < s < t$:

$$h(u,t) = \left(\frac{g(t,u)}{g(t,t)}\right)'_t.$$

The same holds when we suppose $\min(s,t) = t$. Now, the necessary and sufficient condition for equivalence of processes $x(t)$ and $y(t)$ is

$$\int_0^\tau \int_0^\tau \left(\left(\frac{g(t,u)}{g(t,t)}\right)'_t\right)^2 \mathrm{d}t\,\mathrm{d}u < \infty, \quad u \le t.$$

If this condition is satisfied the spectral multiplicity of $x(t)$ and $y(t)$ will be the same and equal to one because the Markov process $y(t)$ has multiplicity one ([2]). The proof is completed.

Theorem 2. *Let $x(t)$, $t \in [a,b] = T$, be a process given by* (2) *where $z(u)$, $u \in [a,b]$, is a Gaussian process such that the function $f(u) = \partial F(u)/\partial u = \partial E z^2(u)/\partial u$ is continuous and $f(u) \neq 0$, for all $t \in T$. If $g(t,t) \neq 0$, for all $t \in T$, and*

$$\frac{1}{f(t)}\left(\frac{g(t,u)}{g(t,t)}\right)'_t \in L^2(f(t)\,\mathrm{d}t \times f(u)\,\mathrm{d}u),$$

i.e.

$$\int_a^b \int_a^b \frac{1}{f(t)} \left(\left(\frac{g(t,u)}{g(t,t)}\right)'_t\right)^2 f(u)\,\mathrm{d}t\,\mathrm{d}u < \infty, \tag{6}$$

then the process $x(t)$ has multiplicity one.

Proof. In a similar way like in previous proof we can show (solving the next integral equation

$$\int_a^{s\wedge t} \left(g(s,u)\frac{g(t,u)}{g(t,t)} - g(s,s)\right) f(u)\,\mathrm{d}u = \int_a^s \int_a^t h(u,v)\, g(s,u)\, f(u)\, f(v)\,\mathrm{d}u\,\mathrm{d}v, \quad s,t \in T,$$

by unknown function $h(u,v)$) that processes $x(t)$ and $y(t) = \int_a^t g(t,t)\,\mathrm{d}z(u)$, $u \le t$, $u,t \in T$, are equivalent processes if and only if

$$\int_a^b \int_a^b \frac{1}{f(t)} \left(\left(\frac{g(t,u)}{g(t,t)}\right)'_t\right)^2 f(u)\,\mathrm{d}t\,\mathrm{d}u < \infty.$$

In this case the spectral multiplicity of $x(t)$ and $y(t)$ is the same and equal to one. The proof is completed.

Note. The statement of the Theorem 1 is valid even we assume that T is an infinite subinterval of $\mathbf{R}$.

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