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EXISTENCE AND UNIQUENESS OF THE SOLUTION FOR LOBACEVSKY'S FUNCTIONAL EQUATION

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The purpose of this paper is to give a theorem for the existence and uniqueness of solution of Lobacevsky's functional equation and to effective find it.

Theorem 1. *Let* $f:\mathbf{R}\to\mathbf{R}$, $f(0)>0$ *and strictly increasing at zero. Then there exists a unique function such that*

(1) $\quad f(x)f(y)=f\big((x+y)/2\big)^2$,

(2) $\quad f(x)$ *is strictly increasing on* $\mathbf{R}$ *if* $f(0)>0$, $0<f(0)<f(1)=0$, $1<a/f(0)$,

(3) $\quad f$ *is continuous function on* $\mathbf{R}$.

At first we assume that the function f exists and, we highlight some properties of this function.

Proposition 1.

(i) *If there exists* $x_0\in\mathbf{R}$ *such that* $f(x_0)=0$, *then* $f(x)=0$ *for any* $x\in\mathbf{R}$.

(ii) $f(x)f(0)>0$, $\operatorname{sgn} f(x)=\operatorname{sgn} f(0)$.

(iii) $f(nx)=f(0)\Big(\dfrac{f(x)}{f(0)}\Big)^n,\ f(-nx)=\big(f(0)\big)^{-1}\Big(\dfrac{f(x)}{f(0)}\Big)^{-n}$ *for any* $n\in\mathbf{N}$,

$$f(kx)=f(0)\Big(\frac{f(x)}{f(0)}\Big)^k,\ k\in\mathbf{Z},\ f(x/2^n)=f(0)\Big(\frac{f(x)}{f(0)}\Big)^{1/2^n},\ f(0)>0.$$

(iv) $f(n)=f(0)\Big(\dfrac{a}{f(0)}\Big)^n,\ f(k)=f(0)\Big(\dfrac{a}{f(0)}\Big)^k,\ f\Big(\dfrac{k}{2^n}\Big)=f(0)\Big(\dfrac{a}{f(0)}\Big)^{1/2^n},\ f(0)>0.$

Proof. From (1) it follows $f(2x_0-x)f(x)=f(x_0)^2=0$, i.e. (i).

We have $f(x)f(0)=f(x/2)^2$, i.e. (ii).

(iii) follows by induction.

In what follows we consider $f(0)>0$, which by (ii) implies $f(x)>0$.

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Definition. [**3**] *The function $f : I \subseteq \mathbf{R} \to \mathbf{R}$ (I - interval) is called strictly increasing (strictly decreasing) at $x_0 \in I$ if there exist $\delta(x_0) > 0$ so that*

$$\operatorname{sgn} \frac{f(x) - f(x_0)}{x - x_0} = 1\,(-1) \text{ for } 0 < x < x_0 < \delta.$$

Proposition 2. [**3**] *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a function. It is strictly monotonic if and only if it is strictly monotonic at every point of I.*

Proposition 3. *The solution f, $f(0) \neq 0$ is strictly monotonic on $\mathbf{R}$ iff it is strictly monotonic at zero.*

Proof. Taking into account the Definition and Proposition 3 the implication $\Rightarrow$ is obvious. From (1) we get

$$\frac{f(x-x_0) - f(0)}{x - x_0} = \frac{1}{2f(x_0)} \, \frac{f(x/2)^2 - f(x_0/2)^2}{\frac{x - x_0}{2}} \quad \text{for any } x, x_0 \in \mathbf{R},\ x \neq x_0, \text{ i.e.}$$

$$\operatorname{sgn} \frac{f(x/2) - f(x_0/2)}{\frac{x - x_0}{2}} = \operatorname{sgn} \frac{f(x-x_0) - f(0)}{x - x_0}\,,$$

since according to (i), $\operatorname{sgn} \dfrac{2f(x_0)}{f(x/2) + f(x_0/2)} = 1$.

By the assumption, f is strictly increasing (strictly decreasing) at zero, then there exists $\delta(0) > 0$ such that $\operatorname{sgn} \dfrac{f(x-x_0)}{x - x_0} = 1\,(-1)$ for $0 < |x - x_0| < \delta$.

Hence $f(x)$ is a strictly increasing function on $\mathbf{R}$.

Now, we shall use the assumption (2) to draw the function f. We choose $n_0 = 1$, $n_1 = 2$, $n_2 = 2^2, \ldots, n_k = 2^k$ and for given $k \in \mathbf{N}$, we find $m_k \in \mathbf{Z}$ such that

$$m_k \le n_k x < m_k + 1,\ \frac{m_k}{n_k} \le x < \frac{m_k + 1}{n_k} \tag{4}$$

and by (2), (iv) and (4), we have

$$f(0)\Big(\frac{a}{f(0)}\Big)^{m_k/n_k} \le f(x) < f(0)\Big(\frac{a}{f(0)}\Big)^{(m_k+1)/n_k}. \tag{5}$$

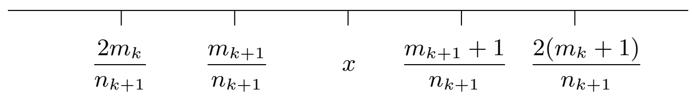
We show that

$$\Big(f(0)\Big(\frac{a}{f(0)}\Big)^{m_k/n_k},\ f(0)\Big(\frac{a}{f(0)}\Big)^{(m_k+1)/n_k}\Big) \supset \Big(f(0)\Big(\frac{a}{f(0)}\Big)^{m_{k+1}/n_{k+1}},\ f(0)\Big(\frac{a}{f(0)}\Big)^{(m_{k+1}+1)/n_{k+1}}\Big). \tag{6}$$

From (4) results

$$2m_k \le 2n_k x = n_{k+1} x < 2\,(m_k + 1), \quad \frac{2m_k}{n_{k+1}} \le x < \frac{2\,(m_k+1)}{n_{k+1}}\,, \quad \ell' = \frac{2}{n_{k+1}}\,, \tag{7}$$

$$\frac{m_{k+1}}{n_{k+1}} \le x < \frac{m_{k+1}+1}{n_{k+1}}, \quad \ell'' = \frac{1}{n_{k+1}} = \frac{1}{2}\,\ell' \tag{8}$$



$$\frac{2m_k}{n_{k+1}} < \frac{m_{k+1}}{n_{k+1}} < \frac{m_{k+1}+1}{n_{k+1}} < \frac{2\,(m_k+1}{n_{k+1}} \tag{9}$$

$$\frac{m_k}{n_k} = \frac{2m_k}{2n_k} = \frac{2m_k}{n_{k+1}} < \frac{m_{k+1}}{n_{k+1}} < \frac{m_{k+1}+1}{n_{k+1}} < \frac{2\,(m_k+1)}{n_{k+1}} = \frac{2\,(m_k+1)}{2n_k} = \frac{m_k+1}{n_k}\,. \tag{10}$$

Proposition 4. [**3**] *Let $b > 1$, $r \in \mathbf{Q}$ and $g(r) = b^r$ be a function. Then $g(r)$ is a strictly increasing and continuous function.*

Taking into account (10) and Proposition 2 results (6). In this way, the intervals

$$\left(f(0)\Big(\frac{a}{f(0)}\Big)^{m_k/n_k},\, f(0)\Big(\frac{a}{f(0)}\Big)^{(m_k+1)/n_k}\right)$$

form a sequence of close and inclusive intervals and, by Cantor's principle, there exists a common point of all intervals and it is unique because the length $\to 0$ when $k \to \infty$, $n_k \to \infty$.

$$\lim_{k\to\infty} \ell_k = \lim\, f(0)\cdot a^{m_k/n_k}\big(a^{1/n_k}-1\big) = 0. \tag{11}$$

We choose for $f(x)$ even the number which corresponds with this point.

In the following we show that the function f satisfies (1). For any $x, y \in \mathbf{R}$ and for given $k \in \mathbf{N}$ corresponds m_k and $p_k \in \mathbf{Z}$ such that

$$\frac{m_k}{n_k} \le x < \frac{m_k+1}{n_k}, \quad \frac{p_k}{n_k} \le y < \frac{p_k+1}{n_k} \tag{12}$$

and

$$\begin{aligned} -f(0)^2\cdot\Big(\frac{a}{f(0)}\Big)^{(m_k+p_k)/n_k}\bigg(\Big(\frac{a}{f(0)}\Big)^{2/n_k}-1\bigg) &\le f(x)f(y) - f\Big(\frac{x+y}{2}\Big)^2 \\ &\le f(0)^2\cdot\Big(\frac{a}{f(0)}\Big)^{(m_k+p_k)/n_k}\bigg(\Big(\frac{a}{f(0)}\Big)^{2/n_k}-1\bigg), \end{aligned} \tag{13}$$

or

$$\bigg|f(x)f(y) - f\Big(\frac{x+y}{2}\Big)^2\bigg| \le f(0)^2\cdot\Big(\frac{a}{f(0)}\Big)^{(m_k+p_k)/n_k}\bigg(\Big(\frac{a}{f(0)}\Big)^{2/n_k}-1\bigg). \tag{14}$$

Taking into account Proposition 2 results ($k \to \infty$, $n_k \to \infty$)

$$f(x)f(y) - f\Big(\frac{x+y}{2}\Big)^2 = 0, \text{ i.e. } (1).$$

Now, we show that the function is strictly increasing on $\mathbf{R}$ if $f(0) > 0$. We assume that $y > x$. From (12) results $p_k/n_k > (m_k+1)/n_k$ and

$$\begin{aligned} f(0)\Big(\frac{a}{f(0)}\Big)^{m_k/n_k} &\le f(x) < f(0)\Big(\frac{a}{f(0)}\Big)^{(m_k+1)/n_k}, \\ f(0)\Big(\frac{a}{f(0)}\Big)^{p_k/n_k} &\le f(y) \le f(0)\Big(\frac{a}{f(0)}\Big)^{(p_k+1)/n_k}. \end{aligned} \tag{15}$$

We have

$$\begin{aligned} 0 &< f(0)\bigg(\Big(\frac{a}{f(0)}\Big)^{p_k/n_k} - \Big(\frac{a}{f(0)}\Big)^{(m_k+1)/n_k}\bigg) < f(y) - f(x) \\ &< f(0)\bigg(\Big(\frac{a}{f(0)}\Big)^{(p_k+1)/n_k} - \Big(\frac{a}{f(0)}\Big)^{m_k/n_k}\bigg), \end{aligned} \tag{16}$$

hence $0 < f(y) - f(x)$ when $y - x > 0$, i.e. (2).

Because

$$\frac{p_k - m_k - 1}{n_k} < y - x < \frac{p_k + 1 - m_k}{n_k} \quad \text{and} \quad \lim_{k\to\infty}(y - x) = 0, \tag{17}$$

from Proposition 4 and (16), results

$$\lim_{k\to\infty}\big(f(y) - f(x)\big) = \lim_{y\to x}\big(f(y) - f(x)\big) = 0, \ \lim_{y\to x} f(y) = f(x), \text{ i.e.} \tag{18}$$

the function f is continuous on $\mathbf{R}$, (3).

Theorem 2. *The function f is differentiable on $\mathbf{R}$ and*

$$f'(0) = \frac{f(0)}{x_0}\,\ln\,\frac{f(x_0)}{f(0)}, \quad x_0 \ne 0, \tag{19}$$

$$f'(x) = \frac{f'(0)}{f(0)}\,f(x), \tag{20}$$

$$f(x) = f(0)\,e^{x_0 f'(0)/f(0)}. \tag{21}$$

Proof. Taking into account (3) and

$$\lim_{n\to\infty} \frac{\Big(\dfrac{f(x_0)}{f(0)}\Big)^{1/2^n} - 1}{1/2^n} = \ln\,\frac{f(x_0)}{f(0)}, \ \ x_0 \ne 0 \tag{22}$$

we have

$$f'(0) = \lim_{n\to\infty} \frac{f\left(\dfrac{x_0}{2^n}\right) - f(0)}{x_0/2^n} = \lim_{x_0/2^n\to 0} \frac{f(0)}{x_0} \cdot \frac{\left(\dfrac{f(x_0)}{f(0)}\right)^{1/2^n} - 1}{1/2^n} = \frac{f(0)}{x_0} \ln \frac{f(x_0)}{f(0)},$$

i.e. (19).

From

$$\frac{f(x-x_0)-f(0)}{x-x_0} = \frac{1}{f(x_0)} \frac{f(x/2)+f(x_0/2)}{2} \frac{f(x/2)-f(x_0/2)}{\dfrac{x-x_0}{2}}$$

we deduce

$$f'(x_0/2) = \lim_{x\to x_0} \frac{f(x/2)-f(x_0/2)}{\dfrac{x-x_0}{2}} = \frac{f'(0)}{f(x_0/2)} f(x_0) - \frac{f'(0)}{f(0)} f(x_0/2)$$

hence $f'(x) = \dfrac{f'(0)}{f(0)} f(x)$ i.e. (20).

From (19) results (21).

Remark. The case $f(0) < 0$ results in $f(x) < 0$, $f : \mathbf{R} \to \mathbf{R} \setminus \{0\}$ and f is strictly decreasing and continuous on $\mathbf{R}$. The case $f(0) = 0$ results in $f(x) = 0$ for any $x \in \mathbf{R}$.

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