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SOME BBP-FUNCTIONS

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Some series in relation to Bailey-Borwein-Plouffe algorithm for π are presented

Our aim is to present a simple proof of the following

Proposition 1. *If* $|z+1| < \sqrt{2}, \quad r \in \mathbf{C}$, *then*

$$\pi + 4\operatorname{arctg} z + (2+8r)\ln\frac{1-2z-z^2}{1+z^2} = (4+8r)\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+1}}{8k+1}\,\frac{1}{16^k} \tag{1}$$

$$-8r\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+2}}{8k+2}\,\frac{1}{16^k} - 4r\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+3}}{8k+3}\,\frac{1}{16^k}$$

$$-(2+8r)\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+4}}{8k+4}\,\frac{1}{16^k} - (1+2r)\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+5}}{8k+5}\,\frac{1}{16^k}$$

$$-(1+2r)\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+6}}{8k+6}\,\frac{1}{16^k} + r\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+7}}{8k+7}\,\frac{1}{16^k}\,.$$

Choosing $r = -1/4$, one finds an interesting development of $\operatorname{arctg} z$. More precisely, we obtain

Corollary 1. *For* $|z+1| < \sqrt{2}$

$$\pi + 4\operatorname{arctg} z = 2\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+1}}{8k+1}\,\frac{1}{16^k} + 2\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+2}}{8k+2}\,\frac{1}{16^k} \tag{2}$$

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$$+\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+3}}{8k+3}\,\frac{1}{16^k}-\frac{1}{2}\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+5}}{8k+5}\,\frac{1}{16^k}$$
$$-\frac{1}{2}\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+6}}{8k+6}\,\frac{1}{16^k}-\frac{1}{4}\sum_{k=0}^{+\infty}\frac{(1+z)^{8k+7}}{8k+7}\,\frac{1}{16^k}\,.$$

At the same time, using in (2) the equality

$$\operatorname{arctg} 1+\operatorname{arctg} z=\operatorname{arctg} t,\qquad t:=\frac{1+z}{1-z}\quad (z<1),$$

we give

Corollary 2. *If t is real such that $|t-1|<\sqrt{2}$, then*

$$\operatorname{arctg} t=\sum_{k=0}^{+\infty}\frac{16^k}{8k+1}\,\frac{t^{8k+1}}{(1+t)^{8k+1}}+2\sum_{k=0}^{+\infty}\frac{16^k}{8k+2}\,\frac{t^{8k+2}}{(1+t)^{8k+2}} \tag{3}$$
$$+2\sum_{k=0}^{+\infty}\frac{16^k}{8k+3}\,\frac{t^{8k+3}}{(1+t)^{8k+3}}-4\sum_{k=0}^{+\infty}\frac{16^k}{8k+5}\,\frac{t^{8k+5}}{(1+t)^{8k+5}}$$
$$-8\sum_{k=0}^{+\infty}\frac{16^k}{8k+6}\,\frac{t^{8k+6}}{(1+t)^{8k+6}}-8\sum_{k=0}^{+\infty}\frac{16^k}{8k+7}\,\frac{t^{8k+7}}{(1+t)^{8k+7}}\,.$$

For instance, if in (1) we select $(z,r)=(0,-1/2)$ and then let $t=0$ in (3), one finds

$$\pi=\sum_{k=0}^{+\infty}\left(\frac{1}{8k+2}+\frac{2}{8k+3}+\frac{2}{8k+4}-\frac{1}{2\,(8k+7)}\right)\frac{1}{16^k},$$
$$\pi=\sum_{k=0}^{+\infty}\left(\frac{2}{8k+1}+\frac{2}{8k+2}+\frac{1}{8k+3}-\frac{1}{2\,(8k+5)}-\frac{1}{2\,(8k+6)}-\frac{1}{4\,(8k+7)}\right)\frac{1}{16^k}.$$

Let us remark that for $(z,r)=(0,0)$ we find from (1) the remarkable BBP formula (attributed to David Bailey, Peter Borwein and Simon Plouffe for π, that is

$$\pi=\sum_{k=0}^{+\infty}\left(\frac{4}{8k+2}-\frac{2}{8k+4}-\frac{1}{8k+5}-\frac{1}{8k+6}\right)\frac{1}{16^k}\,. \tag{4}$$

Also when $(z,r)=(0,r),\ r\in\mathbf{C}$ one finds another formula for π, namely

$$\pi=\sum_{k=0}^{+\infty}\left(\frac{4+8r}{8k+1}-\frac{8r}{8k+2}-\frac{4r}{8k+3}-\frac{2+8r}{8k+4}-\frac{1+2r}{8k+5}\right.$$
$$\left.-\frac{1+2r}{8k+6}+\frac{r}{8k+7}\right)\frac{1}{16^k}\,,$$

which was discovered by Victor Adamchik and Stan Wagon in their interesting HTML paper [**1**].

This is a reason why the function $\mathcal{L}$ defined as

$$\mathcal{L}(z,r) = \pi + 4\arctan z + (2+8r)\ln\frac{1-2z-z^2}{1+z^2} \qquad (|z+1|<\sqrt{2},\ r\in\mathbf{C})$$

may be called a BBP-function.

It is clear that π is expressed as

$$\pi = \mathcal{L}(0,r)$$

for any complex r.

Let us denote

$$\sigma_j(x) = \sum_{k=0}^{+\infty}\frac{x^{8k+j}}{8k+j}\,, \qquad j=1,2,\ldots,\ |x|<1. \tag{5}$$

According to the fact that for $-1<x<1$ we have

$$\sigma_j(x) = \int_0^x \frac{t^{j-1}}{1-t^8}\,\mathrm{d}t,$$

for $j\in\{1,2,\ldots,7\}$ we find that

$$\begin{aligned}8\,\sigma_j(x) = (-1)^{j+1}\ln(1+x) - \ln(1-x) - \pi\sin\frac{j\pi}{4}\cos\frac{j\pi}{2}\\ - \sum_{k=1}^{3}L_k(x)\cos\frac{kj\pi}{4} + 2\sum_{k=1}^{3}A_k(x)\sin\frac{kj\pi}{4}\,,\end{aligned} \tag{6}$$

where

$$L_k(x) = \ln\Big(1-2x\cos\frac{k\pi}{4}+x^2\Big), \qquad A_k(x) = \arctan\frac{x-\cos\frac{k\pi}{4}}{\sin\frac{k\pi}{4}}\,.$$

Lemma 1. *With the above notation, the following equalities hold on* $(-1,1)$

$$\begin{aligned}\sqrt{2}\,\big(\sigma_1(x)+\sigma_3(x)-\sigma_5(x)-\sigma_7(x)\big) &= A_1(x)+A_3(x),\\ 2\,\big(\sigma_2(x)-\sigma_6(x)\big) &= A_1(x)-A_3(x)+\frac{\pi}{2}\,,\\ 4\sqrt{2}\,\big(\sigma_1(x)-\sigma_5(x)\big)-8\,\big(\sigma_4(x)+\sigma_6(x)\big) &= E(x)+L(x),\end{aligned}$$

where the functions $E,\,L:(-1,1)\to\mathbf{R}$ are defined as

$$\boxed{E(x) = \pi + 4\arctan(x\sqrt{2}-1), \qquad L(x) = 2\ln\frac{1-x^2}{1-x\sqrt{2}+x^2}}\,. \tag{7}$$

According to Lemma 1 we have

$$\begin{aligned} E(x) &= 2\sqrt{2}\,\big(\sigma_1(x)+\sigma_3(x)-\sigma_5(x)-\sigma_7(x)\big)+4\,\big(\sigma_2(x)-\sigma_6(x)\big),\\ L(x) &= 2\sqrt{2}\,\big(\sigma_1(x)-\sigma_3(x)-\sigma_5(x)+\sigma_7(x)\big)-4\,\big(\sigma_2(x)+2\sigma_4(x)+\sigma_6(x)\big). \end{aligned} \tag{8}$$

Further let us define $\mathbf{B} : (-1,1)\times\mathbf{C}$ by means of the equality

$$\mathbf{B}(x,r)=E(x)+(1+4r)\,L(x), \qquad (|x|<1,\ r\in\mathbf{C}).$$

Lemma 2. *The function* $\mathbf{B}$ *has the properties*

$$\pi=\mathbf{B}(\sqrt{2}/2,r), \qquad \forall r\in\mathbf{C},$$

$$\begin{aligned} \mathbf{B}(x,r) = {} & 2\sqrt{2}\,(2+4r)\sum_{k=0}^{+\infty}\frac{x^{8k+1}}{8k+1}-16r\lambda\sum_{k=0}^{+\infty}\frac{x^{8k+2}}{8k+2}-8\sqrt{2}\,r\sum_{k=0}^{+\infty}\frac{x^{8k+3}}{8k+3}\\ & -8\,(1+4r)\sum_{k=0}^{+\infty}\frac{x^{8k+4}}{8k+4}-4\sqrt{2}\,(2+2r)\sum_{k=0}^{+\infty}\frac{x^{8k+5}}{8k+5}\\ & -8\,(1+2r)\sum_{k=0}^{+\infty}\frac{x^{8k+6}}{8k+6}+8\sqrt{2}\,r\sum_{k=0}^{+\infty}\frac{x^{8k+7}}{8k+7}\,. \end{aligned}$$

Proof. It is sufficient to use the equalities (5)–(8) and to observe that $L(\sqrt{2}/2)=0$. Now, taking $x=(\sqrt{2}/2)\,(1+z),\ |1+z|<\sqrt{2}$, then from

$$\mathcal{L}(z,r)=\mathbf{B}\Big(\frac{\sqrt{2}\,(1+z)}{2}\,,\,r\Big)$$

we conclude with the proof of the desired equality (1).

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