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SOME RELATIONS AND SUBSETS GENERATED BY PRINCIPAL CONSISTENT SUBSET OF SEMIGROUP WITH APARTNESS

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The investigation is in the Constructive algebra in the sense of E. Bishop, F. Richman, W. Ruitenburg, D. van Dalen and A. S. Troelstra. Algebraic structures with apartness the first were defined and studied by A. Heyting. After that, some authors studied algebraic structures in constructive mathematics as for example: D. van Dalen, E. Bishop, P. T. Johnstone, A. Heyting, R. Mines, J. C. Mulvey, F. Richman, D. A. Romano, W. Ruitenburg and A. Troelstra. This paper is one of articles in their the author tries to investigate semugroups with apartnesses. Relation q on S is a coequality relation on S if it is consistent, symmetric and cotransitive; coequality relation is generalization of apatness. The main subject of this consideration are characterizations of some coequality relations on semigroup S with apartness by means od special ideals $J_{(a)} = \{x \in S : a \# SxS\}$, principal consistent subsets $C_{(a)} = \{x \in S : x \# SaS\}\,(a \in S)$ of S and by filled product of relations on S.

Let $S = (S, =, \neq, \cdot, 1)$ be a semigroup with apartness. As preliminaries we will introduce some special notions, notations and results in set theory, commutative ring theory and semigroup theory in constructive mathematics and we will give proofs of several general theorems in semigroup theory. In the next section we will introduce relation s on S by $(x, y) \in s$ iff $y \in C_{(x)}$ and we will describe internal filfulments $c(s \cup s^{-1})$ and $c(s \cap s^{-1})$ and their classes $A(a) = \cap A_n(a)$ and $K(a) = \cap K_n(a)$ respectively. We will give the proof that the set $K(a)$ is maximal strongly extensional consistent ideal of S for every a in S. Before that, we will analyze semigroup S with relation $q = c(s \cup s^{-1})$ in two special cases: (i) the relation q is a band coequality relation on S : (ii) q is left zero band coequality relation on S. Beside that, we will introduce several compatible equality and coequality relations on S by sets $A(a), A_n(a), K(a)$ and $K_n(a)$.

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0. INTRODUCTION

There are several books on algebraic structures in constructive mathematics in the sense of Bishop, van Dalen, Richman and Ruitenburg as for example:

(i) *Intuitionistic algebra* by W. Ruitenburg (University of Utrecht, 1982);

(ii) *Constructive mathematics* by F. Richman (editor) (Springer lecture notes in mathematics, 873);

(iii) *A course on constructive algebra* by R. Mines, F. Richman and W. Ruitenburg (Springer, 1988);

(iv) *Constructivism in mathematics, An introduction: Volume II* (*Chapter VIII: Algebra*) by A. S. Troelstra and D. van Dalen (North-Holand, 1988).

This author studied constructive set theory ([**3**], [**5**], [**9**], [**10**], [**14**], [**15**] and others), constructive commutative ring theory ([**9**], [**11**], [**12**], [**13**] and others) and constructive semigroup theory ([**6**], [**16**] and several articles forthcoming). This paper is one of articles in it the author tries to investigate semigroup with apartness.

Let $S = (S, =, \neq)$ be a set with apartness. Relation q on S is a coequality relation if it consistent, symmetric and cotransitive. Coequality relation is generalization of apartness. In the part 1.1 of preliminaries we will give several assertions on coequality relation. Besides, we will describe some properties of filled product of relations as for example internel filfullment $c(r)$ of given relation r on S. Filled product of relations the first was defined and studied by the author (1996). We will recall the theorem on existence of maximal coequality relation compatible with given equality relation. Examples I contained some examples of coequality relations. In the part 1.2 we will describe some properties of commutative ring S with apartness. Coideals of commutative ring with apartness the first were defined and studied by W. Ruitenburg (1982). This author proved in his paper [**9**], if T is a coideal of S, then relation q on S, defined by $(x, y) \in q \Leftrightarrow x - y \in T$, is a coequality relation on S compatible with the ring operations. In the Examples II we will show several examples of coideals. Part 1.3 of preliminaries contained notions, notations and some general results on semigroup with apartness: we will introduce notion of cocongruence on semigroup as coequality relation compatible with semigroup operation and we will give several theorems on anticongruence in semigroup with apartness as for example on construction of factor-semigroup. Beside that, we give a construction of anticongruence q^* on semigroup S by given coequality relation q. This anticongruence q^* is minimal extension of q. Finally, we will give proof that the set $C_{(a)} = \{x \in S : x \# SaS\}$ is a consistent subset of S generated by element a of S such that $a \# C_{(a)}$ and we will describe its basic properties.

Section 2 contained the main results of this paper. In the part 2.1 we will describe (Theorem 7 and Theorem 8 and their corollaries) relation $c(s \cup s^{-1})$, its classes $A(a)\,(s \in S)$ and semigroup S in two special cases (Theorem 9 and Theorem 10). In the part 2.2 we will describe relation $q_2 = c(s \cap s^{-1})$ and its classes $K(a)\,(a \in S)$: (i) relation q is anticongruence on S such that the factor-semigroup $S/(\overline{q}, q)$ is semilattice (Theorem 12); (ii) the set $K(a)$ is a maximal strongly extensional consistent ideal of S such that $a \# K(a)$ (Theorem 13 and Corollary 13.0). The section 3 are references.

1. PRELIMINARIES

1.1. Equality and coequality on set with apartness

Let $(X, =, \neq)$ be a set in the sense of BISHOP ([**1**]), MINES ([**6**]), TROELSTRA and VAN DALEN ([**17**]), where $\neq$ is a binary relation on X which satisfies the following properties

$$\neg(x \neq x),\ x \neq y \Rightarrow y \neq x,\ x \neq z \Rightarrow x \neq y \vee y \neq z,\ x \neq y \wedge y = z \Rightarrow x \neq z$$

called *apartness* (A. HEYTING). Let Y be a subset of X and $x \in X$. By $x\#Y$ we denote $(\forall y \in Y)\,(y \neq x)$ and by $\overline{Y}$ we denote subset $\{x \in X : x\#Y\}$ ([**9**], [**14**]). The subset X of Y is *strongly extensional* in X if and only if $y \in Y \Rightarrow y \neq x \vee x \in Y$. Let $f : X \to Y$ be a function of sets with apartnesses. f is *strongly extensional* ([**1**], **17**]) if $f(x) \neq f(y)$ implies $x \neq y$ and f is an *embedding* ([**16**], **17**], [**18**]) if $x \neq y$ implies $f(x) \neq f(y)$. A relation q on X is a *coequality* relation on X ([**3**], [**14**]) if and only if

(1) $(\forall x \in X)\big((x,x)\#q\big)$ (consistent ([**7**])),

(2) $(\forall x, y \in X)\big((x,y) \in q \Rightarrow (x,y) \in q\big)$ (symmetric),

(3) $(\forall x, y, z \in X)\big((x,z) \in q \Rightarrow (y,z) \in q \vee (y,z) \in q\big)$ (cotransitive ([**7**])).

Examples I:

(1) The relation $\neg(=)$ is an apartness on the set $\mathbf{Z}$ of integers.

(2) ([**7**], Theorem II, 3.2). The relation q, defined on the set $\mathbf{Q}^{\mathbf{N}}$ by

$$(f,g) \in q \Leftrightarrow (\exists k \in \mathbf{N})(\exists n \in \mathbf{N})(m \geq n \Rightarrow |f(m) - g(m)| > k^{-1}),$$

is a coequality relation.

(3) ([**7**], page 98–99) A ring R is a local ring if for each $r \in R$, either r or $1 - r$ is a unit, and let M be a module over R. The relation q on M, defined by $(x,x) \in q$ if there exists a homomorphism $f : M \to R$ such that $f(x-y)$ is a unit, is a coequality relation on M.

(4) ([**10**], Theorem 4) Let T be a set and $\boldsymbol{J}$ be a subfamily of $\boldsymbol{P}(T)$ such that $\emptyset \in \boldsymbol{J}$, $A \subseteq B \wedge B \in \boldsymbol{J} \Rightarrow A \in \boldsymbol{J}$, $A \cap B \in \boldsymbol{J} \Rightarrow A \in \boldsymbol{J} \vee B \in \boldsymbol{J}$. If $(X_t)_{t \in T}$ is a family of sets, then the relation q on ΠX_t, defined by $(f,g) \in q \Leftrightarrow \{s \in T : f(s) = g(s)\} \in \boldsymbol{J}$, is a coequality relation on the Cartesian product ΠX_t.

(5) Let $f : \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$ be defined by $f(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \det(\mathbf{a}, \mathbf{b}, \mathbf{c})$. Then the relation q, defined by

$$(\mathbf{a}, \mathbf{b}, \mathbf{c})\, q(\mathbf{x}, \mathbf{y}, \mathbf{z}) \Leftrightarrow f(\mathbf{a}, \mathbf{b}, \mathbf{c}) f(\mathbf{x}, \mathbf{y}, \mathbf{z}) < 0,$$

as a coequality relation on the set of linearly independent vectors.

(6) If a and b are coequality relations on a set X, then the relation $a \cup b$ is a coequality relation on X.

Coequality relation was first defined and studied by M. BOŽIĆ and D. A. ROMANO (1985) in their paper [**3**] as generalization of apartness. After that, coequality relation on set with apartness was studied by the author on several his papers ([**5**], [**10**], [**14**], [**15**]). The first we recall the following theorems:

Theorem 0.1. ([**3**], [**14**]) *Let q be a coequality relation on a set X. Then there the subfamily $X/q = \{aq : a \in X\}$ of $\boldsymbol{P}(X)$ such that*

(4) $(\forall a \in X)(\exists A \in X/q)(a\#A)$,

(5) $(\forall B \in X/q)(\exists b \in X)(b\#B)$,

(6) $(\forall A, B \in X/q)(\exists A \neq B \Rightarrow A \cup B = X)$.

Corollary 0.1.1. *Let q be a coequality relation on a set X and let a be an element of X. The set aq is a strongly extensional subset of X such that $a\#aq$.*

Proof. Let q be a coequality relation, x be an element of aq and let y be an arbitrary element of X. $(a, x) \in q$ implies $(a, y) \in q \vee (y, x) \in q$. Thus $y \in aq \vee y \neq x$ by (1), i.e. the set aq is a strongly extensional subset of X and, by (4), holds $a\#aq$. □

Theorem 0.2. ([**3**], [**14**]) *Let V be a subfamily of $\boldsymbol{P}(X)$ such that it satisfies the conditions* $(4) - (6)$ *of the theorem* 0.1. *Then the relation $q(V)$ on X, defined by*

$$(x, y) \in q(V) \Leftrightarrow (\exists Y \in \boldsymbol{V})(x \in Y \wedge y\#Y),$$

is a coequality on X.

A family V of subsets of X is an *antipartition* on X if and only if V satisfies the conditions (4)–(6). The next theorem shows that if we generate an antipartition by means of a coequality relation q, then the coequality relation generated by the antipartition is simply q again; and similarly if we begin with the coequality relation generated by an antipartition, this relation generates the given antipartition.

Theorem 0.3 ([**15**]) *Let* $C(X) = \{c : c$ is a coequality relation on $X\}$ *and* $P(X) = \{Z : Z$ is an antipartition on $X\}$. *Then $q(X/c) = c$ for every $c \in C(X)$ and $X/q(P) = P$ for every $P \in P(X)$.*

Let X be a set with apartness and let f, g be relations on X. The *filled* ([**5**], [**13**], [**14**]) product of the relation f and the relation g is the relation $g * f$ defined by

$$g * f = \{(x, z) \in X \times X : (\forall y \in X)((x, y) \in f \vee (y, z) \in g)\}.$$

The filled product is associative. The product $g * f$ is nonempty if and only if $D(g) \cup R(f) = X$. For $n\,(\geq 2)$ let ${}^{n}f = f * \cdots * f$ (n factors). Put ${}^{1}f = f$. By $c(f)$ we denote the intersection $\bigcup_{n \in \mathbf{N}} {}^{n}f$. In the following theorem we give a very important property of this intersection.

Theorem 0.4. ([**14**]). *Let f be a relation on a set X with apartness. Then the relation $c(f)$ is a cotransitive relation on X.*

It is easily to see that two following lemmas hold.

Lemma 0.4.1. $c\,(f^{-1}) = c\,(f)^{-1}$.

Proof. (i) Assume that $(x,y) \in {}^n(f^{-1}) \Rightarrow (y,x) \in {}^n f$ for every x, z in X. Then

$$\begin{aligned}(x,y) \in {}^{n+1}(f^{-1}) &\Leftrightarrow (\forall t \in S)((x,t) \in {}^n(f^{-1}) \vee (t,y) \in f^{-1})\\ &\Leftrightarrow (\forall t \in S)((y,t) \in f \vee (t,x) \in {}^n f)\\ &\Leftrightarrow (y,x) \in {}^{n+1} f.\end{aligned}$$

Therefore, by induction, we have $(x,y) \in c\,(f^{-1}) \Rightarrow (x,y) \in c\,(f)^{-1}$.

(ii) Let $(x,y) \in c\,(f)^{-1}$, i.e. let $(y,x) \in c\,(f)$. Then $(y,x) \in c\,((f^{-1})^{-1}) \subseteq c\,(f^{-1})^{-1}$ by (i), and $(x,y) \in c\,(f^{-1})$. So, $c\,(f)^{-1} \subseteq c\,(f^{-1})$. □

Lemma 0.4.2. $f \subseteq g \Rightarrow c\,(f) \subseteq c\,(g)$.

Proof. It is clear that $(x,y) \in c\,(f) \Rightarrow (x,y) \in g$. Suppose that $(x,y) \in c\,(f) \subseteq {}^n f$ implies $(x,y) \in {}^n g$ for every x, y in X. If $(x,y) \in c\,(f)$, then $(x,y) \in {}^{n+1} f$, i.e. then $(\forall t \in X)((x,t) \in {}^n f \vee (t,y) \in f)$. Thus $(x,t) \in {}^n g$ or $(t,y) \in g$ for every t in X by hypothesis. So, $(x,y) \in {}^{n+1} g$. By induction, we have $(x,y) \in c\,(g)$. □

If f is a relation on a set X, the relation $c\,(f)$ is called *internal filfullment* of f. As corollaries of above theorem we have the following results:

Theorem 0.5. ([**14**]) *Let f be a relations on set X. Then the relations $a = c\,((f \cup f^{-1}) \cap \neq)$ and $b = c\,(f \cap f^{-1} \cap \neq)$ are coequality relations on X and holds $b \subseteq a$.*

Corollary 0.5.1. ([**5**], [**14**]) *Let e be an equality relation on a set X with apartness. Then the relation $c\,(\overline{e})$ is a maximal coequality relation on X such that*

$$(\forall x, y, z \in X)((x,y) \in e \wedge (y,z) \in c\,(\overline{e}) \Rightarrow (x,z) \in c\,(\overline{e})).$$

Some more on coequality relations on sets with apartness readers can fined in author's survey paper [**14**].

1.2. Ideals and coideals of ring with apartness

Previous contemplation are motivated by the following facts.

Let $(R, =, \neq, +, 0, \cdot, 1)$ be a commutative ring with apartness. A subset Q of R is a *coideal* of R if

$$0 \# Q,\ -x \in Q \Rightarrow x \in Q,\ x + y \in Q \Rightarrow x \in Q \vee y \in Q,\ xy \in Q \Rightarrow x \in Q \wedge y \in Q.$$

Coideals of commutative ring with apartness were first defined and studied by W. Ruitenburg 1982 ([**17**]). After that, coideals (anti-ideals) studied by A. S. Troelstra and D. van Dalen in their monograph [**18**]. The author proved, in his paper [**9**], if Q is a coideal of a ring R, then the relation q on R, defined by $(x,z) \in q \Leftrightarrow x - y \in Q$, satisfies the following properties:

Theorem 0.6. ([**9**], Proposition 2.5)
(7) *q is a coequality relation on R,*
(8) $(\forall x, y, u, v \in R)((x+u, y+v) \in q \vee (u, v) \in q)$,
(9) $(\forall x, y, u, v \in R)((xu, yv) \Rightarrow (x, y) \in q \vee (u, v) \in q)$.

A relation q on R which satisfies the properties (7)–(9) is called *anticongruence* on R (**9**]). If q is an anticongruence on a ring R, then the set $\{x \in R : (x, 0) \in q\}$ is coideal of R ([**9**]). Let J be an ideal of R and Q is coideal of R Wim Ruitenburg, in his dissertation ([**17**], page 33) first stated a demand that $J \subseteq \neg Q$. This condition is equivalent with the following condition

$$(\forall x, y \in R)(x \in J \wedge y \in Q \Rightarrow x + y \in Q). \tag{10}$$

In this case we say that they are *compatible* ([**9**]) and we can construct the quotient-ring $R/(J, Q)$. Wim Ruitenburg, in his dissertation, first stated question on existence an ideal J or R compatible with given coideal Q and question on existence of a coideal Q of R compatible with given ideal J. If e is a congruence on R, determined by the ideal J, and if q is an anticongruence on R, determined by the coideal Q, then J and Q are compatible if and only if

$$(\forall x, y, z \in R)((x, y) \in e \wedge (y, z) \in q \Rightarrow (x, z) \in q). \tag{11}$$

In this case we say that e and q are *compatible* ([**9**]).

More on constructive commutative ring theory the reader can find in the book [**7**] and in the Johnstone's paper [**4**] and the author's papers [**9**], [**11**], [**12**] and [**13**].

Examples II:

(1) Let $R = (R, =, \neq, +, 0, \cdot, 1)$ be a commutative ring with apartness. Then the sets $\emptyset$ and $R_0 = \{x \in R : x \neq 0\}$ are coideals of R. Let a be an element of the ring R. Then the sets $\mathrm{Ann}\,(a)$ and $\mathrm{Cann}\,(a) = \{x \in R : ax \neq 0\}$ are compatible an ideal and coideal of R.

(2) Let m and $i \in \{1, 2, \ldots, n\}$ be integers. We set $m\mathbf{Z} + i = \{mz + i : z \in \mathbf{Z}\}$. Then the set $\cup\{m\mathbf{Z} + i : i \in \{1, \ldots, m-1\}\}$ is a coideal of the ring $\mathbf{Z}$.

(3) Let K be a Richman field and x be an unknown variable under K. Then the set $C = \{f \in K[x] : f(0) \neq 0\}$ is a coideal of the ring $K[x]$.

(4) Let R be a commutative ring with apartness. Then the set $B = R^n$ be a commutative ring with apartness. For $n \in \mathbf{N}$, the set $M_n = \{f \in B : f(n) \neq 0\}$ is a coideal of B.

(5) ([**9**]) Let R be a local ring. Then the set $M = \{a \in R : (\exists x \in R)(ax = 1)\}$ is a coideal of R.

(6) ([**11**]) Let S be a coideal of a ring and let X be a subset of R. Then the set $[S : X] = \{a \in R : (\exists x \in X)(ax \in S)\}$ is a coideal of R.

(7) ([**12**]) Let $\boldsymbol{H}$ be a nonempty family of inhabited subsets of T. Then the set $S(\boldsymbol{H}) = \{r \in \Pi R_t : (\exists A \in \boldsymbol{H})(A \cap Z(r) \neq \emptyset\}$, where $Z(r) = \{t \in T : r(t) \neq 0\}$, is a coideal of the product ΠR_t.

(8) ([**11**]) Let Q be a coideal of a ring R. Then the set $\mathrm{cr}\,(Q) = \{a \in R : (\forall n \in \mathbf{N})(a^n \in Q)\}$ is a coideal of R. If J is an ideal of R compatible with Q, then $\mathrm{cr}\,(Q)$ is compatible with the radical $r(J)$.

(9) If S and T are coideals of a ring R, then the set $S \cup T$ is a coideal of R.

1.3. Equality and coequality relations on semigroup

Let $S = (S, =, \neq, \cdot, 1)$ be a semigroup with apartness and where the semigroup operation is strongly extensional in the next sense

(12) $$(\forall a, b, x, y \in S)(ay \neq by \Rightarrow a \neq b \wedge xa \neq xb \Rightarrow a \neq b).$$

It is equivalent with the following condition

(13) $$(\forall a, b, x, y \in S)(ax \neq by \Rightarrow a \neq b \wedge x \neq y).$$

A subset T of S is a *consistent subset* of S ([**2**]) (or a *coideal* of S) if and only if

(14) $$(\forall x, y \in S)(xy \in T \Rightarrow x \in T \wedge y \in T).$$

A subset T of S is *completely semiprime* ([**2**]) subset of S if $x^2 \in T \Rightarrow x \in T$ $(x \in S)$, and the T is *completely prime* ([**2**]) subset of S if $xy \in T \Rightarrow x \in T \vee y \in T$ $(x, y \in S)$. The ideal is *completely semiprime* (*completely prime*) *ideal* of S ([**2**]) if it is completely semiprime (completely prime) subset of S. Let T be a consistent subset (coideal) of a semigroup S. T is a *filter* of S if T is a subsemigroup of S ([**2**]); T is a *semifilter* of S if and only if $(\forall x \in S)(x \in T \Rightarrow x^2 \in T)$.

Examples III ([**16**]): Let e be an idempotent of a semigroup S with apartness. Then

(1) $A(e) = \{a \in S : ae \neq a\}$ is a strongly extensional right consistent subset of S.

(2) $B(e) = \{b \in S : eb \neq b\}$ is a strongly extensional left consistent subset of S.

(3) $X(e) = \{a \in S : e \# Sa\}$ is a strongly extensional left ideal of S.

(4) $Y(e) = \{b \in S : e \# bS\}$ is a strongly extensional right ideal of S.

(5) $Z(e) = \{x \in S : e \# SxS\}$ is a strongly extensional ideal of S.

(6) The set $M(e) = A(e) \cup B(e) \cup X(e) \cup Y(e)$ is a strongly extensional completely prime subset of S.

Examples IV: (1) Let $\mathbf{R}$ be a set of reals. Then the set $S = [0, 1]$ is a semigroup with apartness under the usual multiplication. Further, the set $J = [0, 1/2\rangle$ is an ideal of S and the sets $Q = [1/2, 1]$ and $T = \langle 0, 1]$ are consistent subsets of S, and the set T is a filter of S.

(2) Let $S = (S, =, \neq, \cdot, 1)$ be a semigroup with apartness and let a be an element of S. Then the set $I(a) = \{x \in S : a \in SxS\}$ is consistent subset of S. This immediately follows

from the inclusion $SxyS \subseteq SxS \cap SyS$. Besides, the following $a \in I(b) \Leftrightarrow b \in J(a)$ holds. The relation t on S, defined by $(a, b) \in t \Leftrightarrow b \in I(a)$, is a reflexive and transitive relation on S.

(3) The set $J_{(a)} = \{x \in S : a\#SxS\}\,(a \in S)$ is an ideal of a semigroup S with apartness such that $a\#J_{(a)}$.

(4) Let $S = (\mathbf{N}, =, \cdot, 1)$. Then: (a) $J(n) = n\mathbf{N}$; (b) $J_{(n)} = \{n+1, n+2, \ldots\}$; (c) $C_{(n)} = \mathbf{N} \setminus n\mathbf{N}$; (d) $I(n) = \{x \in \mathbf{N} : (\exists y \in \mathbf{N})(n = xy)\}$.

A coequality relation q on a semigroup S with apartness is called *anticongruence* or coequality relation compatible with the semigroup operation on S if and only if

$$(\forall a, b, x, y \in S)((ax, by) \in q \Rightarrow (a, b) \in q \vee (x, y) \in q). \tag{15}$$

We start with the following theorem in which we give a very important property of anticongruence q on a semigroup S.

Theorem 1. *Let q be an anticongruence on a semigroup S with apartness. Then the relation $\overline{q}$ is a congruence on S compatible with q.*

Proof. It is true that $= \subseteq \overline{q}$ and that $\overline{q}$ is symmetric. We need to prove that is transitive. Let $(x, y)\#q$ and $(y, z)\#q$ and let (u, v) be an arbitrary element of q. Then $(u, x) \in q \vee (x, y) \in q \vee (y, z) \in q \vee (z, v) \in q$. From here follows $u \neq x$ or $z \neq v$. So, $(u, v) \neq (x, z)$ and $(x, z)\#q$.

Suppose that $(a, b)\#q$ and $(x, y)\#q$. Let (u, v) be an arbitrary element of q. Then $(u, ax) \in q \vee (ax, by) \in q \vee (by, v) \in q$. Thus we have $u \neq ax$ or by $\neq v$ because $(ax, by) \in q$ implies $(a, b) \in q$ or $(x, y) \in q$ what is impossible. So, $(ax, by)\#q$. □

As corollary of above theorem we can construct the quotient-semigroup $S/(\overline{q}, q) = \{a\overline{q} : a \in S\}$

Theorem 2. *If q is an anticongruence on a semigroup S with apartness, then the set $S/(\overline{q}, q)$ is a semigroup with*

$$a\overline{q} = b\overline{q} \Leftrightarrow (a, b)\#q,\ a\overline{q} \Leftrightarrow (a, b) \in q,\ a\overline{q} \cdot b\overline{q} = ab\overline{q}.$$

Proof. Let $a\overline{q} = x\overline{q}$ and $b\overline{q} = y\overline{q}$, i.e. let $(a, x)\#q$ and $(b, y)\#q$. Let (u, v) be an arbitrary element of q. Then $(u, ab) \in q$ or $(ab, xy) \in q$ or $(xy, v) \in q$. Thence $u \neq ab \vee (a, x) \in q \vee (b, y) \in q \vee xy \neq v$ and, by hypothesis, $(u, v) \neq (ab, xy)$. So, $(ab, xy)\#q$.

Suppose that $ab\overline{q} \neq xy\overline{q}$, i.e. suppose that $(ab, xy) \in q$. Then $(a, x) \in q \vee (b, y) \in q$. Therefore, $a\overline{q} \neq x\overline{q}$ or $b\overline{q} \neq y\overline{q}$. So, the semigroup operation is strongly extensional.

Finally, we have $a\overline{q} \cdot (b\overline{q} \cdot c\overline{q}) = a\overline{q} \cdot bc\overline{q} = a(bc)\overline{q} = (ab)c\overline{q} = ab\overline{q} \cdot c\overline{q} = (a\overline{q} \cdot b\overline{q}) \cdot c\overline{q}$. □

Beside that, we have the statement that the family $S/q = \{aq : a \in S\}$ is a semigroup.

Theorem 3. *Let q be an anticongruence on a semigroup S with apartness. Then the set S/q is a semigroup with*

$$aq = bq \Leftrightarrow (a,b)\#q,\ aq \neq bq \Leftrightarrow (a,b) \in q,\ aq \cdot bq = abq.$$

Proof. Let $aq = xq$ and $bq = yq$. Suppose that $s \in qbq$, i.e. suppose that $(ab,s) \in q$. Then $(ab,xy) \in q$ or $(xy,s) \in q$. Thus, by compatibility of q with the semigroup operation, we have $(a,x) \in q \vee (by) \in q \vee (xy,s) \in q$. So, $s \in xyq$ because $x\#xq = aq$ and $y\#yq = bq$. Therefore, we have $abq \subseteq xyq$. Similarly $xyq \subseteq abq$.

Let $abq \neq xyq$. Then $(ab,xy) \in q$. Thus $(a,x) \in q \vee (b,y) \in q$. So, $aq \neq xq$ or $bq \neq yq$.

Finally, we have $xq \cdot (yq \cdot zq) = xq \cdot (yzq) = x(yz)q = (xy)zq = xyq \cdot zq = (xq \cdot yq) \cdot zq$. □

Corollary 3.0. *Let q be an anticongruence on a semigroup S with apartness. There exists a strongly extensional and embedding isomorphism $\theta : S/(\overline{q},q) \to S/q$.*

Proof. From (i) $a\overline{q} = b\overline{q} \Leftrightarrow (a,b)\#q \Leftrightarrow aq = bq$; (ii) $a\overline{q} \neq b\overline{q} \Leftrightarrow (a,b) \in q \Leftrightarrow aq \neq bq$; (iii) $\theta(a\overline{q} \cdot b\overline{q}) = \theta(ab\overline{q}) = abq = aq \cdot bq$. We conclude that the map θ is strongly extensional and embedding isomorphism of semigroups. □

Theorem 3 implies very interesting corollary:

Corollary 3.1. *Let q be an anticongruence on a semigroup S with apartness. Then the map $p : S \to S/q$, defined by $p(x) = xq\,(x \in S)$, is a strongly extensional epimorphism of semigroups.*

Opposite, we have the following theorem which proof is technically

Theorem 4. *If $f : S \to P$ is a strongly extensional homomorphism of semigroups with apartnesses, then the set $q = \{(x,y) \in S \times S : f(x) \neq f(y)\}$ is an anticongruence on S* (called *cokernel* of f and we denote it by Coker (f)) *such that $q \subseteq \neq$. Further, the relations* Ker (f) *and* Coker (f) *are compatible.*

As corollary of above theorem we have an interesting corollary.

Corollary 4.1. *Let e and q be compatible a congruence and an anticongruence on an semigroup S with apartness. Then there exists a strongly extensional and embedding epimorphism $g : S/(e,q) \to S/(\overline{q},q)$ and there exists a strongly extensional and embedding isomorphism $(S/(e,q))(Ker(g), Coker(g)) \cong S/(\overline{q},q)$.*

This section we will finish with the following two results. By the first we will give a construction of anticongruence on semigroup based on given coequality relation.

Theorem 5. *Let q be a coequality relation on a semigroup S with apartness. Then the relation $q^* = \{(x,y) \in S \times S : (\exists a,b \in S)((axb, ayb) \in q)\}$ is an anticongruence on S such that $q \subseteq q^*$ If s is an anticongruence on S such that $q \subseteq s$, then $q^* \subseteq s$.*

Proof.

$$
\begin{aligned}
(1)\ \ (x,y) \in q* &\Leftrightarrow (\exists a,b \in S)((axb, ayb) \in q)\\
&\Rightarrow (\exists a,b \in S)(\forall u \in S)((axb, ayb) \neq (aub, aub))\\
&\Leftrightarrow (\exists a,b \in S)(\forall u \in S)(axb \neq aub \vee ayb \neq aub)\\
&\Rightarrow (\forall u \in S)(x \neq u \vee y \neq u) \qquad\qquad \text{(by (13))}\\
&\Leftrightarrow (\forall u \in S)((x,y) \neq (u,u)).\\
(x,y) \in q^* &\Leftrightarrow (\exists a,b \in S)((axb, ayb) \in q)\\
&\Leftrightarrow (\exists a,b \in S)((ayb, axb) \in q) \qquad\qquad \text{(by (2)))}\\
&\Leftrightarrow (y,x) \in q^*.\\
(x,z) \in q^* &\Leftrightarrow (\forall a,b \in S)((axb, azb) \in q)\\
&\Rightarrow (\forall y \in S)(\exists a,b \in S)((axb, ayb) \in q \vee (ayb, azb) \in q)\\
&\Rightarrow (\forall y \in S)((x,y) \in q^* \vee (y,z) \in q^*).\\
(xu, yv) \in q^* &\Leftrightarrow (\exists a,b \in S)((axub, ayvb) \in q)\\
&\Rightarrow (\exists a,b \in S)((axub, ayub) \in q \vee (ayub, ayvb) \in q)\\
&\Rightarrow (\exists a, ub \in S)((ax(u,b), ay(ub)) \in q)\\
&\qquad\qquad \vee (\exists ay, b \in S)(((ay)ub), (ay)vb) \in q)\\
&\Rightarrow (x,y) \in q^* \vee (u,v) \in q^*.\\
(2)\ \ (x,y) \in q &\Rightarrow (\exists 1 \in S)((1x1, 1y1) \in q)\\
&\Rightarrow (x,y) \in q^*.
\end{aligned}
$$

(3) Let s be a cocongruence on S such that $q \subseteq s$. Then

$$
\begin{aligned}
(x,y) \in q^* &\Leftrightarrow (\exists a,b \in S)((axb, ayb) \in q)\\
&\Rightarrow (\exists a,b \in S)((axb, ayb) \in s)\\
&\Rightarrow (x,y) \in s \qquad\qquad \text{(by (13))}. \ \square
\end{aligned}
$$

Therefore, the relation q^* is the minimal extension of q.

Semigroup with apartness the first was defined and studied by Heyting. After that, several authors have worked on this important topic as for example Ruitenburg ([**17**]), Troelstra and van Dalen ([**18**]), Johnstone ([**4**]), Mulvey ([**8**]) and the author of this paper ([**6**], [**16**]).

Let $S = (S, =, \neq, \cdot, 1)$ be a semigroup with apartness. As preliminaries we will introduce some special notions, notations and results in set theory, commutative ring theory and semigroup theory in constructive mathematics and we will give proofs of several general theorems in semigroup theory. In the next section we will introduce relation s on S by $(x,y) \in s$ iff $y \in C_{(x)}$ and we will describe internal fulfilments $c\,(s \cup s^{-1})$ and $c\,(s \cap s^{-1})$ and their classes $A(a) = \cap A_n(a)$ and $K(a) = \cap K_n(a)$ respectively. We will give the proof that the set $K(a)$ is a maximal strongly extensional consistent ideal of S for every a in S. Before that, we will analyze semigroup S with relation $q = c\,(s \cup s^{-1})$ in two special cases:

(i) the relation q is a band coequality relation on S; (ii) q is a left zero band coequality relation on S. Beside that, we will introduce several compatible equality and coequality relations on S by sets $A(a)$, $A_n(a)$, $K(a)$ and $K_n(a)$.

For undefined notions we refer to [**1**], [**2**], [**7**], [**9**], [**14**], [**16**], [**17**], [**18**].

1.4. Principal consistent subset

Let S be a semigroup with apartness. We introduce notion of principal consistent subset of S and we introduce some relations defined by these sets. We start with the following theorem:

Theorem 6. *Let a and b be elements of S. The the set $C_{(a)} = \{x \in S : x\#SaS\}$ is a consistent subset of S such that*:

(i) $a\#C_{(a)}$;

(ii) $C_{(a)} \neq \emptyset \Rightarrow 1 \in C_{(a)}$;

(iii) *Let a be an invertible element of S. Then $C_{(a)} = \emptyset$;*

(iv) $(\forall x, y \in S)(C_{(a)} \subseteq C_{(xay)}$;

(v) $C_{(a)} \cup C_{(b)} \subseteq C_{(ab)}$.

Proof.

(0) $$xy \in C_{(a)} \Leftrightarrow xy\#SaS \Rightarrow xy\#SaSy \wedge xy\#xSaS$$
$$\Rightarrow y\#SaS \wedge x\#SaS \Leftrightarrow y \in C_{(a)} \wedge x \in C_{(a)}.$$

(1) Let x be an arbitrary element of $C_{(a)}$. Then $x\#SaS$, and thus $x \neq a$.

(2) Suppose that $C_{(a)} \neq \emptyset$. Then there exists the element x of S such that $x \in C_{(a)}$. Thus, $x \cdot 1 \in C_{(a)}$ and, by (0), we have $1 \in C_{(a)}$.

(3) Let a be an invertible element of S. Then there exists the element b of S such that $ab = 1$. If $C_{(a)} \neq \emptyset$, then, by (2), $1 \in C_{(a)}$. Therefore, $a \in C_{(a)} \wedge b \in C_{(a)}$, what is impossible. So, $C_{(a)} = \emptyset$.

(4) Let x, y be arbitrary element of S and let $u\#SaS$. Then $u\#SxayS$. Therefore, $C_{(a)} \subseteq C_{(xay)}$.

(5) From (4) immediately follows $C_{(a)} \subseteq C_{(ab)} \wedge C_{(b)} \subseteq C_{(ab)} \Rightarrow C_{(a)} \cup C_{(b)} \subseteq C_{(ab)}$. □

If by G_1 we will denote the subgroup of all invertible elements of S, we will have:

Corollary 6.1. *Let a be an element of a semigroup S with apartness such that $C_{(a)} \neq \emptyset$. Then $G_1 \subseteq C_{(a)}$.*

Proof. $x \in G_1 \Rightarrow (\exists y \in S)(xy = 1 \in C_{(a)}) \Rightarrow x \in C_{(a)}$. □

Corollary 6.2. *Let a and b be elements of a semigroup S with apartness such that $C_{(a)} \neq \emptyset$ and $C_{(b)} \neq \emptyset$. Then $C_{(a)} \cap C_{(b)} \neq \emptyset$.*

Let a be an arbitrary element of a semigroup S with apartness. The consistent subset (coideal) $C_{(a)}$ is called a *principal* consistent subset (*principal* coideal) of S generated by a such that $a\#C_{(a)}$.

NOTE. Note that if we get $(a,b) \in q$ if and only if $C_{(a)} \cup C_{(b)} = S$, then we will have

$$C_{(a)} \neq C_{(b)} \Leftrightarrow (\exists x \in C_{(a)})(x \# C_{(b)}) \vee (\exists y \in C_{(b)})(y \# C_{(a)})$$

and

$$\begin{aligned} xy \in S = C_{(a)} \cup C_{(b)} &\Rightarrow xy \in C_{(a)} \vee xy \in C_{(b)} \\ &\Rightarrow (x \in C_{(a)} \wedge y \in C_{(a)}) \vee (x \in C_{(a)} \wedge y \in C_{(b)}) \end{aligned}$$

what is impossible. So, it is not correct definition of a coequality relation.

Examples V: (1) The relation α on S, defined by $(a,b) \in \alpha \Leftrightarrow C_{(a)} = C_{(b)}$ is an equality on S and the relation $c(\overline{\alpha})$ is a coequality relation on S compatible with α.

(2) Let β be the GREEN relation on S, defined by $(a,b) \in \beta \Leftrightarrow SaS = SbS$. Then the relation $c(\overline{\beta})$ is a coequality relation on S compatible with β. Clearly that hold $\beta \subseteq \alpha$ and, by lema 0.4.2, $c(\overline{\alpha}) \subset c(\overline{\beta})$.

(3) Let a and b be elements of a semigroup S with apartness. Then the set $J_{(a)} = \{x \in S : a \# SxS\}$ is an ideal of S such that $a \# J_{(a)}$ and $J_{(ab)} \subseteq J_{(a)} \cap J_{(b)}$. The relation γ on S, defined by $(a,b) \in \gamma \Leftrightarrow J_{(a)} = J_{(b)}$, is an equality relation on S compatible with the coequality relation $c(\overline{\gamma})$.

(4) If we define $(a,b) \in \delta$ by $I(a) = I(b)$, then δ is an equality relation $c(\overline{\delta})$ is a maximal coequality relation on S compatible with δ. We do not know what kind of interrelation exists between relations δ and γ.

2. COEQUALITY RELATIONS GENERATED BY $C_{(a)}$

2.1. Relation $c(s \cup s^{-1})$

In this section we introduce relation s, defined by

$$(a,b) \in s \Leftrightarrow b \in C_{(a)}$$

and we will describe some properties of relations s, $s \cup s^{-1}$ and $q = c(s \cup s^{-1})$. It is clear that $(a,b) \in s \Leftrightarrow a \in J_{(b)}$. We start with some descriptions of relation s.

Theorem 7. *The relation s has the following properties:*

(vi) s *is a consistent relation*;

(vii) $(a,b) \in s \Rightarrow (\forall x,y \in S)((xay,b) \in s)$;

(viii) $(a,b) \in s \Rightarrow (\forall n \in \mathbf{N})((a^n,b) \in s)$;

(ix) $(\forall x,y \in S)((a,xby) \in s \Rightarrow (a,b) \in s)$;

(x) $(\forall x,y \in S)\neg((a,xay) \in s)$.

Proof. (7) Let $(a,b) \in s$, i.e. let $b \in C_{(a)}$ and let x,y be arbitrary elements of S. Then $b \in C_{(xay)}$ by (iv). So, by definition of s, we have $(xay,b) \in s$.

(9) Let $(a, xby) \in s$ for some a, b, x, y in S. Then $xby \in C_{(a)}$ and, by (0), $b \in C_{(a)}$, i.e. $(a, b) \in s$.

(10) Suppose that $(a, xay) \in s$. Then, by (vii), we have $(xay, xay) \in s$ what is impossible by (0). So, for every elements x and y of $S \neg((a, xay) \in s)$ holds. □

Corollary 7.1. *The relation $s \cup s^{-1}$ is a consistent and symmetric relation on S.*

By Theorem 0.4 we can construct, symmetric and cotransitive relation $(s \cup s^{-1})$. As corollary of this theorem we have the following results:

Corollary 7.2. *The relation $c\,(s \cup s^{-1})$ is a coequality relation on S.*

Corollary 7.3. *The relation $c\,(s)$ is a consistent and cotransitive relation on a semigroup S with apartness and the set $q = c\,(s) \cup c\,(s)^{-1}$ is a coequality relation on S and $c\,(s) \cup c\,(s)^{-1} \subseteq c\,(s \cup s^{-1})$ holds.*

For an element a of a semigroup S and for $n \in \mathbf{N}$ we introduce the following notations

$$A_n(a) = \{x \in S : (a, x) \in^n (s \cup s^{-1})\}, \quad A(a) = \{x \in S : (a, x) \in c\,(s \cup s^{-1})\}.$$

By the following results we will present some basic characteristics of these sets.

Theorem 8. *Let a and b be elements of a semigroup S. Then:*

(xi) $A_1(a) = \{x \in S : x\#SaS \vee a\#SxS\} = C_{(a)} \cup J_{(a)}$;

(xii) $A_{n+1}(a) \subseteq A_n(a)$;

(xiii) $A_{n+1}(a) = \{x \in S : S = A_n(a) \cup A_1(x)\}$;

(xiv) $A(a) = \bigcap_{n \in \mathbf{N}} A_n(a)$;

(xv) $a\#A(a)$;

(xvi) *The set $A(a)$ is a strongly extensional subset of S such that $a\#A(a)$.*

Proof. Put $h = s \cup s^{-1}$.

(12)–(14) Let x be an arbitrary element of $A_{n+1}(a)$. Then for every $t \in S$ we have $(a, t) \in^n h \vee (t, x) \in h$. So, $t \in A_n(a) \cup A_1(x)$, i.e. $S = A_n(a) \cup A_1(x)$. As $(x, x)\#h$, we have $(a, x) \in^n h$, i.e. $x \in A_n(a)$.

(15) If x is an element of $A(a)$, then $(a, x) \in c\,(h)$. Thus $(a, x) \neq (a, a)$ because the relation $c\,(h)$ is consistent. Therefore, $x \neq a$. □

Corollary 8.1. *Let $n \in \mathbf{N}$. The relation β_m on a semigroup S with apartness, defined by $(a, b) \in \beta_n \Leftrightarrow A_n(a) = A_n(b)$ is an equality relation on S and the relation $c\,(\overline{\beta_n})$ is a coequality relation on S compatible with β_n.*

Corollary 8.2. $(\forall n \in \mathbf{N})(\beta_n \subseteq \beta_{n+1})$.

Proof. Let $(x, y) \in \beta_n$, i.e. let $A_n(x) = A_n(y)$. Suppose that $v \in A_{n+1}(x)$. Then $S = A_n(x) \cup A_1(v) = A_n(y) \cup A_1(v)$. So, $v \in A_{n+1}(y)$. Therefore $A_{n+1}(x) \subseteq A_{n+1}(y)$. Analogously, we have $A_{n+1}(y) \subseteq A_{n+1}(x)$. Hence $(x, y) \in \beta_{n+1}$. □

Corollary 8.3. $(\forall n \in \mathbf{N})(c\,(\overline{\beta_{n+1}}) \subseteq c\,(\overline{\beta_n}))$.

Corollary 8.4. *The relation β^∞ on a semigroup S with apartness, defined by $(a,b) \in \beta^\infty \Leftrightarrow A(a) = A(b)$, is an equality relation on S and the relation $c\,(\overline{\beta^\infty})$ is a coequality relation on S compatible with β^∞, and $\bigcap\limits_{n \in \mathbf{N}} \beta_n \subseteq \beta^\infty$ holds.*

Corollary 8.5. *Let a be an element of S. The relation $q(a)$ on S, defined by $(x,y) \in q(a)$ iff $x \neq y \wedge (x \in A(a) \vee y \in A(a))$ is a coequality relation on S such that*

$$x \in A(a) \Rightarrow xq = \{y \in S : y \neq x\},\ x\#A(a) \Rightarrow xq = A(a).$$

Proof. (i) It is clear that $q(a)$ is consistent and symmetric relation. Let $(x,z) \in q(a)$, i.e. let $x \neq z$ and $x \in A(a) \vee z \in A(a)$. If y is an element of S, we have, for example:

$$\begin{aligned} x \neq z \wedge x \in A(a) &\Rightarrow (x \neq y \vee y \neq z) \wedge x \in A(a) \\ &\Rightarrow (x \neq y \wedge x \in A(a)) \vee (y \neq z \wedge x \in A(a) \wedge (x \neq y \vee y \in A(a)) \\ &\Rightarrow (x,y) \in q(a) \vee (y,x) \in q(a). \end{aligned}$$

(ii) If $x \in A(a)$, then it is clearly that $xq(a) = \{y \in S : y \neq x\}$ Let $x\#A(a)$, i.e. let $x \neq y$ for every $y \in A(a)$. Then $xq(a) = A(a)$. □

For coequality relation q on a semigroup S with apartness we say that it is a band coequality relation iff $(\forall a \in S)((a,a^2)\#q)$. In the following theorems we will describe semigroup S in which q is a band coequality relation.

Theorem 9. *Let S be a semigroup with apartness. Then the following conditions are equivalent:*

(9.1) *relation q is a band coequality relation on S;*

(9.2) *for every a in S the set $A(a)$ is a completely semiprime subset of S such that $x \in A(a) \Rightarrow x^2 \in A(a)$;*

(9.3) *for every a in S $A(a) = A(a^2)$ holds.*

Proof. (1) $\Rightarrow$ (3). Let q be a band coequality relation on S. If $x \in A(a)$, i.e. if $(a,x) \in q$ we have $(a,a^2) \in q \vee (a^2,x) \in q$ and $x \in A(a^2)$ because $(a,a^2)\#q$. So, $A(a) \subseteq A(a^2)$. Opposite inclusion we prove analogously. Therefore, $A(a) = A(a^2)$.

(3) $\Rightarrow$ (2). Let $x^2 \in A(a)$, i.e. let $(a,x^2) \in q$. Then $(a,x) \in q \vee (x,x^2) \in q$. Thus $x \in A(a)$, because $(x,x^2) \in q$ give $x^2 \in A(x) = A(x^2)$ what is impossible by (xv). At the other hand, we have sequence implications $x \in A(a) \Leftrightarrow (a,x) \in q \Rightarrow (a,x^2) \in q \vee (x^2,x) \in q \Rightarrow x^2 \in A(a)$ because $x \in A(x^2) = A(x)$ is impossible. Therefore, the set $A(a)$ is a completely semiprime subset of S such that $x \in A(a) \Rightarrow x^2 \in A(a)$ for every $a \in S$.

(2) $\Rightarrow$ (1). Let (u,v) be an arbitrary element of q and let a be an element of S. Then $(u,a) \in q \vee (a,a^2) \in q \vee (a^2,v) \in q$. Thus, $u \neq a \vee a \neq v \vee a^2 \in A(a) = A(a^2)$. Hence $(u,v) \neq (a,a^2)$. So, the relation q is a band coequality relation on S. □

For coequality relation c on S we say that it is left zero band coequality relation iff $(a, ab)\#c$ for every elements a and b of S. In the following theorem we will describe semigroup S when the relation q is that relation.

Theorem 10. *Let S be a semigroup with apartness. Then the following conditions are equivalent*:

(10.1) *for every element a in S $A(a) = A(ab)$ holds*;

(10.2) $(a, ab)\#q$ *for every elements a, b in S*;

(10.3) *for every element a in S the set $A(a)$ is a left consistent right ideal of S.*

Proof. (1) $\Rightarrow$ (2). Let (u, v) be an arbitrary element of q and let a, b be elements of S. Then $(u, a) \in q \vee (a, ab) \in q \vee (ab, v) \in q$. Thus $u \neq a \vee ab \neq v \vee ab \in A(a) = A(ab)$. Hence $(u, v) \neq (a, ab)$ because $ab\#A(ab)$. So, $(a, ab)\#q$.

(2) $\Rightarrow$ (3). Let $xy \in A(a)$, i.e. let $(a, xy) \in q$. Then $(a, x) \in q \vee (x, xy) \in q$ and $x \in A(a)$ because $(x, xy)\#q$. So, the set $A(a)$ is a left consistent subset of S. Further, if $x \in A(a)$, i.e. if $(a, x) \in q$, we have $(a, xy) \in q \vee (xy, x)$. Thus $xy \in A(a)$. Therefore, the set $A(a)$ is a right ideal of S.

(3) $\Rightarrow$ (1).

$$\begin{aligned} x \in A(a) &\Leftrightarrow a \in A(x) & \qquad y \in A(ab) &\Leftrightarrow ab \in A(y) \\ &\Rightarrow ab \in A(x) & &\Rightarrow a \in A(y) \\ &\Leftrightarrow x \in A(ab). & &\Leftrightarrow y \in A(a). \end{aligned}$$

So, we have $A(ab) = A(a)$. □

2.2. Relation $c(s \cap s^{-1})$

In this section we will describe relations $s \cap s^{-1}$ and $c\,(s \cap s^{-1})$.

Theorem 11. *The relation $h = s \cap s^{-1}$ has the following properties*:

(xvii) *h is a consistent relation on S*;

(xviii) *h is a symmetric relation on S*;

(xix) $(\forall a, x \in S)\Big(\neg\big((a, xa) \in h\big) \wedge \neg\big((xa, a) \in h\big)\Big)$;

(xx) $(\forall a, y \in S)\Big(\neg\big((ay, a) \in h\big) \wedge \neg\big((a, ay) \in h\big)\Big)$;

(xxi) $(\forall a \in S)(\forall n \in \mathbf{N})\Big(\neg\big((a^n, a) \in h\big) \wedge \neg\big((a, a^n) \in h\big)\Big)$.

Proof. (17) and (18) implies immediately by definition of h and because s is a consistent relation.

(19)–(20) Let x and a be elements of a semigroup S such that $(a, xa) \in s \cap s^{-1}$. Then $(xa, a) \in s \wedge (a, xa) \in s$, what is impossible by (x). Assertion (20) we proved analogously.

(21) This assertion follows immediately from (xix). □

Theorem 12. *The relation $c\,(h)$ is an anticongruence on S such that*

(xxii) $(\forall a, b, x, y \in S)\big((a, b) \in c\,(h) \Leftrightarrow (xay, xby) \in c\,(h)\big)$.

(xxiii) $(\forall a \in S)(\forall n \in \mathbf{N})\big((a, a^n)\#c\,(h) \wedge (a^n, a)\#c\,(h)\big)$.

(xxiv) $(\forall a, b \in S)\big((ab, ba)\# c\,(h)\big)$.

Proof. It is clearly that $c\,(h)$ is a coequality relation on S.

$$\begin{aligned}(xa, xb) \in c\,(h) &\Rightarrow (xa, a) \in c\,(h) \vee (a, b) \in c\,(h) \vee (b, xb) \in c\,(h)\\ &\Rightarrow (a, b) \in c\,(h) \qquad \text{by (xix) and (xx))}\\ &\Rightarrow (a, xa) \in c\,(h) \vee (xa, xb) \in (xb, b) \in c\,(h)\\ &\Rightarrow (xa, xb) \in c\,(h).\end{aligned}$$

The equivalency $(ay, by) \in c\,(h) \Leftrightarrow (a, b)$ is proved analogously.

Therefore, the relation $c\,(h)$ is an anticongruence on semigroup S.

Let a, b be elements of S such that $(a, b) \in c\,(h)$ and let x, y be elements of S. Then $(a, xa) \in c\,(h) \vee (xa, xay) \in c\,(h) \vee (xay, xby) \in c\,(h) \vee (xby, xb) \in c\,(h) \vee (xb, b) \in c\,(h)$. Thus, by (xix) and (xx), we have $(xay, xby) \in c\,(h)$.

Let (u, v) be an arbitrary element of $c\,(h)$, $a, b \in S$ and $n \in \mathbf{N}$. We have

$$\begin{aligned}(u, v) \in c\,(h) &\Rightarrow (u, a^n) \in c\,(h) \vee (a^n, a) \in c\,(h) \vee (a, v) \in c\,(h)\\ &\Rightarrow u \neq a^n \vee a \neq v\\ &\Rightarrow (u, v) \neq (a^n, a).\end{aligned}$$

Besides, we have

$$\begin{aligned}(u, v) \in c\,(h) &\Rightarrow (u, ab) \in c\,(h) \vee (ab, a) \in c\,(h) \vee (a, ba) \in c\,(h) \vee (ba, v) \in c\,(h)\\ &\Rightarrow u \neq ab \vee ba \neq v\\ &\Leftrightarrow (u, v) \neq (ab, ba). \ \square\end{aligned}$$

Corollary 12.1. *The semigroup* $S/\big(\overline{c\,(h)}, c\,(h)\big)$ *is a semilattice.*

Let us introduce the following notations, for $a \in S$ and $n \in \mathbf{N}$,

$$K_n(a) = \{x \in S : (a, x) \in^n (s \cap s^{-1})\},\ K(a) = \{x \in S : (a, x) \in c\,(s \cap s)\}.$$

For these sets we have:

Theorem 13. *Let a be an element of a semigroup S and $n \in \mathbf{N}$. Then:*
(xxiv) $K_1(a) = \{x \in S : x\# SaS \wedge a\# SxS\} = C_{(a)} \cap J_{(a)}$;
(xxv) $K_{n+1}(a) \subseteq K_n(a)$;
(xxvi) $K_{n+1}(a) = \{x \in S : S = K_n(a) \cup K_1(x)\}$;
(xxvii) $K(a) = \bigcap K_n(a)$;
(xxviii) $a\# K(a)$;
(xxix) *The set $K(a)$ is a strongly extensional consistent ideal of S*;
(xxx) $K(a^n) = K(a)$.

Proof. We proof only the assertions (xxix).

Clearly is that the set $K(a)$ is a strongly extensional subset of S. Further, we have:

$$\begin{aligned}
xy \in K(a) &\Rightarrow (a, xy) \in c(h) \\
&\Rightarrow \big((a,x)\in c(h) \vee (x,xy)\in c(h)\big)\wedge\big((a,y)\in c(h)\vee(y,xy)\in c(h)\big) \\
&\Rightarrow x \in K(a) \wedge y \in K(a); \\
x\in K(a) \wedge u,v\in S &\Rightarrow (a,x)\in c(h) \wedge u,v \in S \\
&\Rightarrow (a,uxv) \in c(h) \vee (uxv,ux) \in c(h) \vee (ux,x) \in c(h) \\
&\Rightarrow uxv \in K(a).
\end{aligned}$$

So, the set $K(a)$ is a strongly extensional consistent ideal of S such that $a\#K(a)$. □

Corollary 13.0. *The set $K(a)$ is a maximal strongly extensional ideal of S such that $a\#K(a)$.*

Proof. Let T be a strongly extensional consistent ideal of S such that $a\#T$, let t be an arbitrary element of T. Then $a\#StS$ and $t \neq uav \vee uav \in T$ for every $u,v \in S$ because T is strongly extensional subset of S. Thus $uav \in S$ implies that $a \in T$ what is impossible. So, $t\#SaS$. Therefore, $T \subseteq K_1(a)$. Assume that $T \subseteq K_n(a)$. Let t be an arbitrary element of T, z be an arbitrary element of S. Thus $t \neq uzv \vee uzv \in T$ and $xty \neq z \vee z \in T$ for every $u,v,x,y \in S$. Hence $t\#SzS \vee z \in T$ and $z\#StS \vee z \in T$. This means that $z \in K(t)$ or $z \in T \subseteq K_n(a)$. So, $S = K_1(t) \cup K_n(a)$, i.e. $t \in K_{n+1}(a)$. So, $T \subseteq K_{n+1}(a)$. By induction we have that $T \subseteq K_n(a)$ for every $n \in \mathbf{N}$. Thus $T \subseteq \bigcap_{n\in\mathbf{N}} K_n(a) = K(a)$. □

Corollary 13.1. *The relation $c(h)$ is a maximal semilattice anticongruence on S.*

Corollary 13.2. *Let $a \in S$ be an element of a semigroup S. Then the relation $q(a)$, defined by*

$$(x,y) \in q(a) \Leftrightarrow x \neq y \wedge \big(x \in K(a) \vee y \in K(a)\big),$$

is an anticongruence on S such that

$$x \in K(a) \Rightarrow xq(a) = \{y \in S : y \neq x\},\ x\#K(a) \Rightarrow xq(a) = K(a).$$

Corollary 13.3. *Let a be an element of a semigroup S. Then the relation $e(a)$, defined by*

$$(x,y) \in e(a) \Leftrightarrow x = y \vee \big(x \in K(a) \wedge y \in K(a)\big),$$

is a congruence on S such that

$$x \in K(a) \Rightarrow xe(a) = K(a), \quad x\#K(a) \Rightarrow xe(a) = \{x\}$$

compatible with $q(a)$.

Corollary 13.4. *The relation r_n on S, defined by $(a,b) \in r_n \Leftrightarrow K_n(a) = K_n(b)$, is an equality relation on S and the relation $c\left(\overline{r_n}\right)$ is a coequality relation on S compatible with r_n.*

Corollary 13.5 *The relation r^∞ on S, defined by $(a,b) \in r^\infty \Leftrightarrow K(a) = K(b)$, is an equality relation on S and the relation $c\left(\overline{r^\infty}\right)$ is a coequality relation on S compatible with r^∞ and $\bigcap\limits_{n\in\mathbf{N}} r_n \subseteq r^\infty$ holds.*

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