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DETERMINANTS OF MATRICES ON PARTIALLY ORDERED SETS*

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This note extends the results on determinants of greatest common divisor matrices to partially ordered sets.

1. INTRODUCTION

Let $S = \{x_1, x_2, \ldots, x_n\}$ be a set of distinct positive integers. The matrix (S) having the greatest common divisor $(x_i,\ x_j)$ of x_i and x_j as its (i, j)-entry is called the greatest common divisor (GCD for short) matrix on S. The study of GCD matrix was motivated by the work of SMITH, BESLIN and LIGH. SMITH [**2**] showed that the determinant of GCD matrix (S) on a factor-closed set is the product $\phi(x_1)\phi(x_2)\cdots\phi(x_n)$, where ϕ is EULER's totient function. The set S is factor-closed if it contains every divisor of x for any $x \in S$. BESLIN and LIGH [**1**] showed that every GCD matrix is positive definite, and in fact, is the product of a specified matrix and its transpose. It follows from these results that (S) is invertible. BOURQUE and LIGH [**3**] have recently obtained a formula for the inverse of GCD matrix on a factor-closed set.

Our aim in this paper is to extend the above results to partially ordered sets. We first establish a formula for a determinant of a matrix on a partially ordered set, then provide a formula for the inverse of this matrix if it is invertible. As an immediate consequence, the determinant of GCD matrix (S) on a nearly factor-closed set can be easily calculated. The set S is nearly factor-closed if it is not factor-closed but we can add exactly one element, say x_0, to S such that $S \cup \{x_0\}$ is factor-closed.

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2. DETERMINANTS OF MATRIX ON A PARTIALLY ORDERED SET

Let $(S, \leq)$ be a finite partially ordered set, and $f(x)$ be any function on S. Define the $n \times n$ matrices $F = (f_{ij})$ and $G = (g_{ij})$ as follows:

$$f_{ij} = \begin{cases} f(x_i) & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g_{ij} = \sum_{\substack{x_k \leq x_i \\ x_k \leq x_j}} f(x_k).$$

Theorem 1. $\det G = f(x_1)f(x_2)\cdots f(x_n)$.

Proof. Define the Zeta function of the partial order $\leq$ as the $n \times n$ matrix $Z = (z_{ij})$, where

$$z_{ij} = \begin{cases} 1 & \text{if } x_i \leq x_j, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$g_{ij} = \sum_{\substack{x_k \leq x_i \\ x_k \leq x_j}} f(x_k) = \sum_{k=1}^{n} z_{ki} f_{kk} z_{kj} = \sum_{k=1}^{n}\sum_{s=1}^{n} z_{ki} f_{ks} z_{sj},$$

i.e., $G = Z^T F Z$.

Note that the rearrangement of the elements in S doesn't influence on the determinants of Z and G. We can choose an order of $x_1, x_2, \ldots, x_n$ such that Z is a triangular matrix and the entries on the main diagonal are all 1's. Hence we have $\det Z = 1$, and

$$\det G = \det(Z^T F Z) = (\det Z)^2 \det F = \det F = f(x_1)f(x_2)\cdots f(x_n).$$

Theorem 2. *Suppose that $f(x_i) \neq 0$ for all $i = 1, 2, \ldots, n$. Then G is invertible and if we denote the inverse of G by $A = (a_{ij})$, we have*

$$a_{ij} = \sum_{\substack{x_i \leq x_k \\ x_j \leq x_k}} \frac{\mu(x_i, x_k)\mu(x_j, x_k)}{f(x_k)},$$

where $\mu(x, y)$ is the Möbius *function.*

Proof. It follows from Theorem 1 that $\det G = f(x_1)f(x_2)\cdots f(x_n) \neq 0$. Hence G is invertible. Let $U = (u_{ij})$ be the $n \times n$ matrix with $u_{ij} = \mu(x_i, x_j)$. It is a routine exercise to show that $Z^{-1} = U$, which implies that

$$G^{-1} = (Z^T F Z)^{-1} = Z^{-1} F^{-1} (Z^{-1})^T = U F^{-1} U^T,$$

i.e.,

$$a_{ij} = \sum_{\substack{x_i \le x_k \\ x_j \le x_k}} \frac{\mu(x_i, x_k)\mu(x_j, x_k)}{f(x_k)}.$$

Theorem 3. *Suppose that $f(x_i) \neq 0$ for all $i = 1, 2, \ldots, n$. Let $G_{i,j}$ be the matrix obtained by deleting row i and column j of G. Then*

$$\det G_{j,i} = (-1)^{i+j} f(x_1)f(x_2)\cdots f(x_n) \sum_{\substack{x_i \le x_k \\ x_j \le x_k}} \frac{\mu(x_i, x_k)\mu(x_j, x_k)}{f(x_k)}.$$

In particular, we have

$$\det G_{i,i} = f(x_1)f(x_2)\cdots f(x_n) \sum_{k=1}^{n} \frac{\mu^2(x_i, x_k)}{f(x_k)}.$$

Proof. By Theorem 2, G^{-1} exists. So we get that the (i, j)-entry of G^{-1} is equal to

$$(-1)^{i+j} \frac{\det G_{j,i}}{\det G}.$$

By Theorem 2 again, this entry is also equal to

$$\sum_{\substack{x_i \le x_k \\ x_j \le x_k}} \frac{\mu(x_i, x_k)\mu(x_j, x_k)}{f(x_k)}.$$

Note that $\det G = f(x_1)f(x_2)\cdots f(x_n)$ by Theorem 1. We have

$$\det G_{j,i} = (-1)^{i+j} f(x_1)f(x_2)\cdots f(x_n) \sum_{\substack{x_i \le x_k \\ x_j \le x_k}} \frac{\mu(x_i, x_k)\mu(x_j, x_k)}{f(x_k)}.$$

In particular, since $\mu(x_i, x_k) = 0$ if x_i is not $\le x_k$, we have

$$\det G_{i,i} = f(x_1)f(x_2)\cdots f(x_n) \sum_{i=1}^{n} \frac{\mu^2(x_i, x_k)}{f(x_k)}.$$

3. DETERMINANT OF GCD MATRIX

Now we turn to consider the determinant of a GCD matrix (S). We have

Corollary 4. [**1,2**]. *If S is factor-closed, then*

$$\det(S) = \phi(x_1)\phi(x_2)\cdots\phi(x_n).$$

Proof. Define the partial order $\le$ as follows: for x_i, $x_j \in S$, $x_i \le x_j$ means that $x_i \mid x_j$. Take $f(x) = \phi(x)$. By the definition of G, we have

$$g_{ij} = \sum_{\substack{x_k \le x_i \\ x_k \le x_j}} f(x_k) = \sum_{x_k|(x_i,x_j)} \phi(x_k) = \sum_{d|(x_i,x_j)} \phi(d) = (x_i, x_j),$$

where $\sum_{x_k|(x_i,x_j)} \phi(x_k) = \sum_{d|(x_i,x_j)} \phi(d)$ holds because S is factor-closed. Hence $G = (S)$. By Theorem 1, we have $\det(S) = \phi(x_1)\phi(x_2)\cdots\phi(x_n)$.

Define the same partial order as in Corollary 4. By Theorems 2 and 3, we have Corollaries 5 and 6 respectively.

Corollary 5. [**3**]. *If S is factor-closed, then (S) is invertible, and if we denote the (i,j)-entry of the inverse of (S) by a_{ij}, then*

$$a_{ij} = \sum_{\substack{x_i \mid x_k \\ x_j \mid x_k}} \frac{\mu(x_k/x_i)\mu(x_k/x_j)}{\phi(x_k)}.$$

Corollary 6. *If S is factor-closed, $S_t = S\backslash\{x_t\}$ with $x_t \in S$, then*

$$\det(S_t) = \phi(x_1)\,\phi(x_2)\cdots\phi(x_n)\sum_{k=1}^{n}\frac{\mu^2(x_k/x_t)}{\phi(x_k)}.$$

In order to investigate the determinant of GCD matrix on a nearly factor-closed set, we express Corollary 6 in another version.

Corollary 7. *If S is nearly factor-closed, and $S \cup \{x_0\}$ is factor-closed, then*

$$\det(S) = \phi(x_0)\,\phi(x_1)\,\phi(x_2)\cdots\phi(x_n)\sum_{k=0}^{n}\frac{\mu^2(x_k/x_t)}{\phi(x_k)}.$$

Example 1. Let $S = \{p, p^2, \ldots, p^n\}$, where p is prime. It is easy to see that S is nearly factor-closed and $S \cup \{1\}$ is factor-closed. By Corollary 7, we have

$$\det(S) = \phi(1)\,\phi(p)\,\phi(p^2)\cdots\phi(p^n)\sum_{k=0}^{n}\frac{\mu^2(p^k)}{\phi(p^k)}.$$

Since $\mu(p^k) = 0$ if k is greater than 1, we have

$$\begin{aligned}\det(S) &= p^0p^1\cdots p^{n-1}\left(\mu^2(1) + \frac{\mu^2(p)}{\phi(p)}\right)\\ &= p^0p^1\cdots p^{n-1n-1}\left(1 + \frac{(-1)^2}{p-1}\right)\\ &= p^{(n^2-n+2)/2}/(p-1).\end{aligned}$$

Let $D(s,d,n)=\{s,s+d,s+2d,\ldots,s+(n-1)d\}$, where $(s,d)=1$. BESLIN and LIGH [**1**] asked what the value of the determinant of GCD matrix defined on $D(s,d,n)$ is. This question may be very difficult in general, here we make some discussion in some special cases.

EXAMPLE 2. $D(2,1,n)$ is nearly factor-closed, and $\{1\}\cup D(2,1,n)$ is factor-closed. It follows from Corollary 7 that

$$\det\big(D(2,1,n)\big)=\phi(1)\,\phi(2)\,\phi(3)\cdots\phi(n+1)\sum_{k=1}^{n}\frac{\mu^2(k)}{\phi(k)}.$$

Similarly,

$$\det\big(D(3,2,n)\big)=\phi(1)\,\phi(3)\,\phi(5)\cdots\phi(2n+1)\sum_{k=1}^{n+1}\frac{\mu^2(2k-1)}{\phi(2k-1)}.$$

EXAMPLE 3. $D(s,d,n)$ is a progression of primes $\{p_1,p_2,\ldots,p_n\}$ with $n\le p_1+1$, e.g., $\{37,73,109\}$. Clearly $D(s,d,n)$ is nearly factor-closed and $\{1\}\cup D(s,d,n)$ is factor-closed. By Corollary 7, we have

$$\begin{aligned}\det\big(D(s,d,n)\big)&=\phi(1)\,\phi(p_1)\,\phi(p_2)\cdots\phi(p_n)\left(1+\sum_{k=1}^{n}\frac{\mu^2(p_k)}{\phi(p_k)}\right)\\&=(p_1-1)(p_2-1)\cdots(p_n-1)\left(1+\sum_{k=1}^{n}\frac{1}{p_k-1}\right).\end{aligned}$$

Note that

$$D(s,d,n)=\begin{pmatrix}p_1&1&\cdots&1\\1&p_2&\cdots&1\\\vdots&\vdots&\ddots&\vdots\\1&1&\cdots&p_n\end{pmatrix}.$$

As a verification, it is easy to see that

$$\begin{aligned}\det\big(D(s,d,n)\big)&=\det\begin{pmatrix}1&1&\cdots&1\\0&&&\\\vdots&&D(S,d,n)&\\0&&&\end{pmatrix}\\&=\det\begin{pmatrix}1&1&1&\cdots&1\\-1&p_1-1&0&\cdots&0\\-1&0&p_2-1&\cdots&\vdots\\\vdots&\vdots&\cdots&\ddots&0\\-1&0&\cdots&0&p_n-1\end{pmatrix}\end{aligned}$$

$$= \det \begin{pmatrix} 1+\sum\limits_{k=1}^{n} \frac{1}{p_k-1} & 1 & 1 & \cdots & 1 \\ 0 & p_1-1 & 0 & \cdots & 0 \\ 0 & 0 & p_2-1 & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & p_n-1 \end{pmatrix}$$

$$= (p_1-1)(p_2-1)\cdots(p_n-1)\left(1+\sum_{k=1}^{n}\frac{1}{p_k-1}\right),$$

as desired.

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