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AN APPLICATION OF A GENERALIZED KKM PRINCIPLE ON THE EXISTENCE OF AN EQUILIBRIUM POINT

Zoran D. Mitrović

Using some results by S. PARK we derive a new sufficient condition for the existence of an equilibrium point in the economic model of supply and demand.

1. INTRODUCTION

The classical KNASTER–KURATOWSKI–MAZURKIEWICZ theorem has numerous applications in various fields of mathematics. KY FAN generalized the KKM theorem to subsets of any topological vector space. Since then, various applications and generalizations of FAN'S result have been obtained. This research area is now called KKM theory. The KKM theory is the study of KKM maps and their applications, [**4**].

In this paper, using some results by S. PARK, [**3**] we present results on existence of an equilibrium point for a convex space.

2. PRELIMINARIES

Let A be a subset of a topological vector space X. Let $P(A)$ be the family of all nonempty subsets of A. If A is nonempty set, $\operatorname{co}(A)$ denotes the convex hull of A. Let

$$\Delta_0 = \{p = (p_1, \ldots, p_n) \in \mathbf{R}^n : \sum_{i=1}^{n} p_i = 0\}, \tag{1}$$

$$\Delta_1 = \{p = (p_1, \ldots, p_n) \in \mathbf{R}^n : p_i \geq 0,\ \sum_{i=1}^{n} p_i = 1\}, \tag{2}$$

$$J_p = \{k : p_k \neq 0\},\ \ p \in \Delta_1 . \tag{3}$$

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The following economic model of supply and demand is given in [**2**]:
Let $D, S : \mathbf{R}_+^n \backslash \{0\} \to \mathbf{R}_+^n$, where

$$\mathbf{R}_+^n = \{x = (x_1, \dots, x_2) \in \mathbf{R}^n : x_i \geq 0,\ i = 1, \dots, n\}.$$

The function of supply and demand $\xi : \mathbf{R}_+^n \backslash \{0\} \to \mathbf{R}^n$ is defined as $\xi(p) = D(p) - S(p)$ for each $p \in \mathbf{R}_+^n \backslash \{0\}$.

The economic interpretation is as follows: x is a vector of available goods in an economy, D is a vector of demand, S is a vector of supply. An equilibrium means that the demand equals supply. A point p is called an equilibrium point if $\xi(p) = 0$.

In this paper we present a result which holds in a general convex space, not necessarily finite dimensional. A convex space X is a nonempty convex set X (in a vector space) with any topology which induce the Euclidean topology on the convex hulls of its finite subsets.

For the subset A of a topological space Y we say that it is compactly closed (compactly open) in Y if $A \cap K$ is closed (open) in K, for each compact set $K \subset Y$. By $C(X, Y)$ we denote the family of all continuous mappings from X to Y. For a family of sets $\{A_i : i \in I\}$ we say that it has the finite intersection property if $\bigcap\{A_j : j \in J\} \neq \emptyset$, for each finite set $J \subset I$. Let E be a vector space and $X \subset E$ an arbitrary subset. A map $G : X \to P(E)$ is called a Knaster-Kuratowski-Mazurkiewicz map or simply a KKM-map provided

$$\mathrm{co}\{x_1, \dots, x_n\} \subset \bigcup_{i=1}^{n} G(x_i)$$

for each finite subset $\{x_1, \dots, x_n\}$ of X.

Proposition 1. [**3**] *Let X be a convex space and $D \subset X$ a nonempty set. Further, let be a $F : D \to P(Y)$, where Y is a topological space, and let $s \in C(X, Y)$. Suppose that the following conditions hold :*
(i) for each $x \in D, F(x)$ is compactly closed in Y, or
(i') for each $x \in D, F(x)$ is compactly open in Y,
(ii) for each finite subset $\{x_1, \dots, x_n\}$ of D, $s(\mathrm{co}\{x_1, \dots, x_n\}) \subset \bigcup_{i=1}^{n} F(x_i)$.

Then the family $\{F(x) : x \in D\}$ has the finite intersection property.

3. MAIN RESULTS

The following theorem is a generalization of Theorem 1 from [**2**].

Theorem 1. *Let $\xi \in C(\Delta_1, \Delta_0)$ and $s \in C(\Delta_1)$. Suppose that*

$$(\forall\ p \in \Delta_1)\ (\forall\ k \in \{1, \dots, n\})\ p_k = 0 \Rightarrow \xi_k(s(p)) \geq 0.$$

Then there exists a $\widehat{p} \in \Delta_1$, such that $\xi(\widehat{p}) = 0$.

Proof. For each $i \in I = \{1, \ldots, n\}$ let us define a set $F_i \subset \Delta_1$ by

$$F_i = \{p \in \Delta_1 : \xi_i(p) \leq 0\}.$$

We shall prove that all the conditions of Proposition 1 (with condition (i)) are satisfied. Since ξ_i is continuous and $\xi(p) \in \Delta_0$ for all $p \in \Delta_1$, F_i is closed and nonempty subset of Δ_1 for each $i \in I$. For each nonempty subset $J \in I$ we have to prove that

$$s(\mathrm{co}\{e_j : j \in J\}) \subset \bigcup_{j \in J} F_j.$$

This means that if $y \in s(\mathrm{co}\{e_j : j \in J\})$ then there is a $p \in \Delta_1$ such that $y = s(p)$, i.e. $p = \sum\limits_{j \in J} \lambda_j e_j$ for some $0 \leq \lambda_j \leq 1$ for all $j \in J$. Let $s(p) \notin F_j$ for each $j \in J$. Then $\xi_j(s(p)) > 0$ for each $j \in J$ and $\sum\limits_{j=1}^{n} \xi_j(s(p)) > 0$ because $\xi_j(s(p)) \geq 0$ for each $j \notin J$. This is a contradiction since $\xi(s(p)) \in \Delta_0$. Hence, $y \in \bigcup\limits_{j \in J} F_j$. Because of Proposition 1, $\bigcap\limits_{i \in I} F_i \neq \emptyset$, i.e. there exists $\overline{p} \in \Delta_1$, such that $\xi_i(s(\overline{p})) \leq 0$ for each $i \in I$ and $\sum\limits_{i \in I} \xi_i(s(\overline{p})) = 0$. Therefore $\xi_i(s(\overline{p})) = 0$ for each $i \in I$. Now it is enough to put $\widehat{p} = s(\overline{p})$.

Corollary 1. [**2**] *Let $\xi \in C(\Delta_1, \Delta_0)$. Suppose that*

$$(\forall\, p \in \Delta_1)\ (\forall\, k \in \{1, \ldots, n\})\ p_k = 0 \Rightarrow \xi_k(p) \geq 0.$$

Then there exists a $\widehat{p} \in \Delta_1$ such that $\xi(\widehat{p}) = 0$.

Proof. The proof follows from Theorem 1, if we put $s = id_{\Delta_1}$. □

The following example shows that Theorem 1 is not a consequence of the results by W. K. Kim and D. I. Rim [**2**].

Example. Let

$$\Delta_1 = \{(p_1, p_2) : p_1,\ p_2 \geq 0,\ p_1 + p_2 = 1\},$$
$$\Delta_0 = \{(p_1, p_2) : p_1 + p_2 = 0\},$$

$\xi \in C(\Delta_1, \Delta_0), \quad \xi(p_1, p_2) = (\frac{1}{2} + p_1 - p_2, -\frac{1}{2} - p_1 + p_2).$

Then the Corollary 1 can not be applied, $(\xi_1(0,1) = -\frac{1}{2} \leq 0)$, but we can apply the Theorem 1 with $s(p_1, p_2) = (0, 1)$.

Theorem 2. *Let $\xi_1 : \Delta_1 \to \mathbf{R}^n$ be a continuous mapping and for some $s \in C(\Delta_1)$ and some $\epsilon > 0$ that the following conditions holds*:
(i) $(\forall\, i \in \{1, \ldots, n\})$ $(\exists p \in \Delta_1)$, so that $\xi_i(s(p)) < \epsilon$,
$(ii)$$(\forall\, p \in \Delta_1)$ $(\exists\, i_0 \in J_p)$, so that $\xi_{i_0}(s(p)) < \epsilon$,
(iii) if $\frac{1}{n} \sum\limits_{i=1}^{n} \xi_k(s(p)) < \epsilon$ then $\xi(s(p)) = 0$.
Then there exists $\widehat{p} \in \Delta_1$, so that $\xi(\widehat{p}) = 0$.

Proof. For each $i \in I = \{1, \ldots, n\}$ let us define set $G_i \subset \Delta_1$ by

$$G_i = \{p \in \Delta_1 : \xi_i(s(p)) < \epsilon\}.$$

Because of (i) each G_i is nonempty. Using the continuity of ξ and s, we conclude that each G_i is open.

Let J be nonempty subset of I. For each $p \in \mathrm{co}\{e_j : j \in J\}$ there exists λ_j with $0 \leq \lambda_j \leq 1$ for each $j \in J$, so that $p = \sum_{j \in J} \lambda_j e_j$. Because of (ii)

$$s(\mathrm{co}\{e_j : j \in J\}) \subset \bigcup_{j \in J} G_j.$$

Now because of Proposition 1 (with condition (ii)) we have that $\bigcap_{i=1}^{n} G_i \neq \emptyset$, so there exists $\overline{p} \in \bigcap_{i=1}^{n} G_i$, and $\xi_i(s(\overline{p})) < \epsilon$ for all $i \in \{1, \ldots, n\}$. Since $\sum_{i=1}^{n} \xi_i(s(\overline{p})) < n\epsilon$, because of (iii) we have $\xi(s(\overline{p})) = 0$. Now if we put $\widehat{p} = s(\overline{p})$ the proof is finished. □

Corollary.2. *Let $\xi : \Delta_1 \to \mathbf{R}^n$ be a continuous mapping and let there exists $\epsilon > 0$, so that the following conditions holds :*
(i) $(\forall\, i \in \{1, ..., n\})$ $(\exists p \in \Delta_1)$, so that $\xi_i(p) < \epsilon$,
$(ii)(\forall\, p \in \Delta_1)$ $(\exists\, i_0 \in J_p)$, so that $\xi_{i_0}(p) < \epsilon$,
(iii) if $\frac{1}{n} \sum_{i=1}^{n} \xi_k(p) < \epsilon$, then $\xi(p) = 0$.
Then there exists $\widehat{p} \in \Delta_1$, so that $\xi(\widehat{p}) = 0$.

Proof. The proof follows from Theorem 2 with $s = id_{\Delta_1}$. □

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Faculty of Electrical Engineering,
University of Banjaluka,
Banjaluka,
Republika Srpska

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