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ON THE CONVERGENCE OF THE SERIES $\sum a_n^{1-x_n/n}$

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We show that, for any sequence (a_n) of positive numbers and any bounded sequence (x_n) of real numbers, the series $\sum a_n$ and $\sum a_n^{1-x_n/n}$ either both converge or both diverge.

Throughout this paper, the letters **N** and **R** will stand for the sets of all natural and real numbers, respectively. We start with a useful inequality.

Lemma. *If* $a, x, \delta \in \mathbf{R}$ *and* $n \in \mathbf{N}$ *such that* $0 < a \leq 1$ *and* $|x| \leq \delta \leq n$, *then*

$$a^{1-x/n} < (a + 2^{-n})\, 2^{\delta}.$$

Proof. If $x < 0$, then $1 < 1 - x/n$. Hence, since $0 < a \leq 1$ and $0 \leq \delta$, it follows that

$$a^{1-x/n} \leq a \leq a\, 2^{\delta}.$$

Suppose now that $0 \leq x$. If $a < 2^{-n}$, then since $0 \leq 1 - x/n$ and $x \leq \delta$ it is clear that

$$a^{1-x/n} \leq \left(2^{-n}\right)^{1-x/n} = 2^{-n}\, 2^{x} \leq 2^{-n}\, 2^{\delta}.$$

While, if $2^{-n} \leq a$, then $a^{-1/n} \leq 2$. Hence, since $0 \leq x \leq \delta$ and $0 < a$, it follows that

$$a^{1-x/n} = a \left(a^{-1/n}\right)^{x} \leq a\, 2^{x} \leq a\, 2^{\delta}.$$

Therefore, the required inequality is also true.

Now, by using the above lemma, we can easily prove the following

Theorem. *Let* (a_n) *be a sequence in* **R** *such that* $a_n > 0$ *for all* $n \in \mathbf{N}$. *Then the following assertions are equivalent*:

(1) *the series* $\sum a_n$ *converges*;

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(2) *the series* $\sum a_n^{1-x_n/n}$ *converges for all bounded sequence* (x_n) *in* $\mathbf{N}$;

(3) *the series* $\sum a_n^{1-x_n/n}$ *converges for some bounded sequence* (x_n) *in* $\mathbf{R}$.

Proof. Suppose that the assertion (1) holds and (x_n) is a bounded sequence in $\mathbf{R}$. Then $(a_n) \to 0$ and $\delta = \sup_{n\in\mathbf{N}} |\, x_n| < +\infty$. Therefore, there exists $n_0 \geq \delta$ such that $a_n \leq 1$ for all $n \geq n_0$. Now, by the above lemma, it is clear that

$$a_n^{1-x_n/n} \leq (a_n + 2^{-n})\, 2^{\delta}$$

for all $n \geq n_0$. Hence, since the series $\sum a_n$ and $\sum 2^{-n}$ converge, it follows that the series $\sum a_n^{1-x_n/n}$ also converges.

Since the implication (2) $\Rightarrow$ (3) is trivially true, suppose now that the assertion (3) holds. Define $\delta = \sup_{n\in\mathbf{N}} |x_n|$ and choose $n_0 \in \mathbf{R}$ such that $1+\delta \leq n_0$. Then, for all $n \geq n_0$, we have

$$1 \leq n_0 - \delta \leq n - \delta \leq n - |x_n| \leq n - x_n \leq |n - x_n|.$$

Therefore, we may define a sequence (y_n) in $\mathbf{R}$ such that

$$y_n = n\, x_n/(x_n - n)$$

for all $n \geq n_0$. Then, by the triangle inequality, it is clear that

$$|y_n| = |x_n - x_n^2/(n - x_n)| \leq |x_n| + |x_n|^2/|n - x_n| \leq \delta + \delta^2$$

for all $n \geq n_0$. Therefore, the sequence (y_n) is bounded. Hence, by the implication (1) $\Rightarrow$ (2), it follows that the series $\sum\, (a_n^{1-x_n/n})^{1-y_n/n}$ converges. Now, since

$$a_n = (a_n^{1-x_n/n})^{1-y_n/n}$$

for all $n \geq n_0$, it is clear that the assertion (1) also holds.

The following example shows that the assumption that the sequence (x_n) is bounded cannot be dropped or even weakened to the assumption that (x_n/n) is a null sequence.

Example. Let (a_n) and (x_n) be sequences in $\mathbf{R}$ such that $a_1 > 0$ and

$$a_n = \frac{1}{n\big(\log(n)\big)^2} \quad \text{and} \quad x_n = \frac{n}{1+\sqrt{\log\,\big(n\log(n)\big)}}$$

for all $n \geq 2$. Then the series $\sum a_n$ converges, but the series $\sum a_n^{1-x_n/n}$ diverges despite that $(x_n/n) \to 0$.

By using Cauchy's condensation test, it can be easily shown that the series $\sum a_n$ converges, but the series $\sum a_n \log(n)$ diverges [**2**, p. 399]. Therefore, to prove the divergence of the series $\sum a_n^{1-x_n/n}$, it is enough to show only that

$$a_n \log(n) \leq a_n^{1-x_n/n}$$

for all $n \geq 3$. For this, assume that $n \geq 3$ and define

$$q_n = \sqrt{\log\,\big(n \log(n)\big)}\,.$$

Then, by using that $e \leq n$ and the functions log and sqrt are increasing, we can easily see that $1 \leq \log(n)$, $\log(n) \leq \log(n\log(n))$, and hence $\sqrt{\log(n)} \leq q_n$. Hence, since $\log(x) \leq \sqrt{x}$ for all $x > 0$, we can infer that

$$\log\,\big(\log(n)\big) \leq q_n.$$

This implies that $q_n \log\big(\log(n)\big) \leq q_n^2$. Therefore, we also have

$$\big(\log(n)\big)^{q_n} = e^{q_n \log(\log(n))} \leq e^{q_n^2} = e^{\log(n\log(n))} = n\log(n).$$

This implies that $(\log(n))^{1+q_n} \leq n\,(\log(n))^2 = a_n^{-1}$. Therefore, we also have

$$\log(n) \leq \big(a_n^{-1}\big)^{1/(1+q_n)} = a_n^{-1/(1+q_n)}.$$

Hence, it follows that

$$a_n \log(n) \leq a_n^{1-1/(1+q_n)} = a_n^{1-x_n/n}.$$

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