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L^p–APPROXIMATION OF SOLUTIONS OF STOCHASTIC INTEGRODIFFERENTIAL EQUATIONS

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This paper is concerned with the construction of the approximate solution of the general stochastic integrodifferential equation of the Ito type, defined on a partition of the time-interval. The closeness of the original and approximate solutions is measured in the sense of the L^p-norm.

1. INTRODUCTION

In many fields of science and engineering there is a large number of problems which are intrinsically nonlinear and complex in nature, involving stochastic excitations of a Gaussian white noise type. Having in mind that a Gaussian white noise is an abstraction and not a physical process, mathematically described as a formal derivative of a Brownian motion process, all such problems are mathematically modelled by stochastic differential equations, or in more complicated cases, by stochastic integrodifferential equations of the Ito type [**5**]. Since these equations are not solvable in most cases, it is important to find their approximate solutions in an explicit form, or in a form suitable for applications of numerical methods.

Throughout the paper let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space on which all random variables and processes are defined. For notational simplicity reason, we shall omit $\omega \in \Omega$ in all random functions and we shall restrict ourselves to one-dimensional case – the multidimensional case is analogous and is not difficult in itself.

We consider a stochastic process $x = (x_t, t \in [0,1])$, defined as a solution of the following stochastic integrodifferential equation of the Ito type

$$(1)\qquad \mathrm{d}x_t = F\Big(t, x_t, \int_0^t f_1(t,s,x_s)\,\mathrm{d}s, \int_0^t f_2(t,s,x_s)\,dw_s\Big)\,\mathrm{d}t$$

$$+G\Big(t, x_t, \int_0^t g_1(t,s,x_s)\,\mathrm{d}s, \int_0^t g_2(t,s,x_s)\,\mathrm{d}w_s\Big)\,\mathrm{d}w_t,\ \ t \in [0,1],$$

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$$x_0 = x(0). \ \text{a.s.},$$

in which $w = (w_t, t \geq 0)$ is a normalized Brownian motion with a natural filtration $\{\mathcal{F}_t, t \geq 0\}$ (i.e. $\mathcal{F}_t = \sigma\{w_s, 0 \leq s \leq t\}$), and x_0 is a random variable independent of w. The random functions $f_i : J \times R \times \Omega \to R$, $g_i : J \times R \times \Omega \to R$, $i = 1, 2$, $F : [0,1] \times R^3 \times \Omega \to R$ and $G : [0,1] \times R^3 \times \Omega \to R$, where $J = \{(t,s) : 0 \leq s \leq t \leq 1\}$, are BOREL measurable on their domains, $f_i(t,s,x)$ and $g_i(t,s,x)$ are $\mathcal{F}_s$-measurable for each $s \leq t, x \in R$, $F(t,x,y,z)$ and $G(t,x,y,z)$ are $\mathcal{F}_t$-measurable for each $(x,y,z) \in R^3$. The stochastic process x is a strong solution of Eq. (1), i.e. it is adapted to $\{\mathcal{F}_t, t \geq 0\}$, $x_o = x(0)$ a.s., all LEBESGUE's and ITO's integrals in the integral form of Eq. (1) are well defined, and Eq. (1) is satisfied almost surely for all $t \in [0,1]$.

Note that Eq. (1) contains the more general stochastic differential and integral equations as special cases, earlier studied by many authors in the literature, in many papers by MURGE and PACHPATTE [**8**], for example. On the basis of the classical theory of stochastic differential equations of the ITO type, one can prove the basic existence and uniqueness theorem, based on the PICARD method of iterations: Let $E|x_0|^2 < \infty$ and the random functions f_i, g_i, F and G be globally Lipschitzian and satisfy the usual linear growth condition, i.e. let there exist a constant $L > 0$ such that, for all $(t,s) \in J$ and $(x,y,z), (x',y',z') \in R^3$, with probability one,

$$\begin{aligned} |F(t,x,y,z) - F(t,x',y',z')| &\leq L\,(|x-x'| + |y-y'| + |z-z'|), \\ |f_i(t,s,x) - f_i(t,s,x')| &\leq L\,|x-x'|, \quad i = 1,2, \end{aligned} \tag{2}$$

$$\begin{aligned} |F(t,x,y,z)|^2 &\leq L^2(1 + |x|^2 + |y|^2 + |z|^2), \\ |f_i(t,s,x)|^2 &\leq L^2(1 + |x|^2), \quad i = 1,2, \end{aligned} \tag{3}$$

and analogously for G, g_1, g_2. Then Eq. (1) has a unique a.s. continuous strong solution x satisfying $E\{\sup_{t\in[0,1]} |x_t|^2\} < \infty$. Moreover, by applying the procedure used in [**7**], one can prove that if $E|x_0|^p < \infty$ for any number $p > 0$, then $E\{\sup_{t\in[0,1]} |x_t|^p\} < \infty$.

There is a number of papers in which the solution of the stochastic differential equation $\mathrm{d}x_t = a(t,x_t)\,\mathrm{d}t + b(t,x_t)\,\mathrm{d}w_t$, $t \in [0,1]$, $x_0 = x(0)$, is approximated on partitions $\Gamma_n, n \in \mathbf{N}$

$$0 = t_0 < t_1 < \cdots < t_n = 1, \quad \delta_n = \max_{0\leq k\leq n-1} (t_{k+1} - t_k), \tag{4}$$

of the interval $[0,1]$. For example, in paper [**6**] the solution x is approximated by the solutions $x^n, n \in \mathbf{N}$ of the equations $\mathrm{d}x_t^n = a(t_k, x_{t_k}^n)\,\mathrm{d}t + b(t_k, x_{t_k}^n)\,\mathrm{d}w_t$, $t \in [t_k, t_{k+1})$, $0 \leq k \leq n-1$, $x_0^n = x_0$, in the sense of the L^p-norm, $p \geq 2$. The rate of this closeness is $O(\delta_n^{1/2})$ when $n \to \infty$. This result has earlier been obtained in [**3**] for $p = 2$. In the present paper we shall compare in the L^p-norm, under more general conditions than in [**6**], the solution of Eq. (1) by the solutions of the corresponding equations of the same type, defined on a partitions (4) of the interval $[0,1]$. The treatment used in our analysis is partially inspired by the treatment used earlier in paper [**1**] for stochastic differential equations of the Ito type.

2. MAIN RESULTS

Let (4) be a partition Γ_n of the interval $[0,1]$ and x^n be the solution of the equation

$$\begin{aligned}\mathrm{d}x_t^n &= F\Big(t, x_{t_k}^n, \int_{t_k}^t f_1(t,s,x_{t_k}^n)\,\mathrm{d}s, \int_{t_k}^t f_2(t,s,x_{t_k}^n)\,\mathrm{d}w_s\Big)\,\mathrm{d}t\\ &+G\Big(t, x_{t_k}^n, \int_{t_k}^t g_1(t,s,x_{t_k}^n)\,\mathrm{d}s, \int_{t_k}^t g_2(t,s,x_{t_k}^n)\,\mathrm{d}w_s\Big)\,\mathrm{d}w_t,\\ &\qquad t\in[t_k,t_{k+1}),\quad 0\le k\le n-1,\\ x_{t_0}^n &= x_0 \ \text{ a.s.},\quad x_{t_k}^n = x^n(t_k-0)\ \text{ a.s.},\ 1\le k\le n-1.\end{aligned} \tag{5}$$

In fact, the solution $x^n = (x_t^n,\, t\in[0,1])$ is constructed as an a.s. continuous process, by attaching successively processes $(x_t^n, t\in[t_k,t_{k+1}])$, $0\le k\le n-1$, on the points $t_k, 1\le k\le n-1$ of the partition Γ_n.

The main goal of this paper is to show that x^n is an approximate solution to the solution x of Eq. (1), in the sense of the L^p-norm, $p\ge 2$.

For notational simplicity reason, let us denote that

$$\begin{aligned}(Fx_k)(t) &= F\Big(t, x_t, \int_{t_k}^t f_1(t,s,x_s)\,\mathrm{d}s, \int_{t_k}^t f_2(t,s,x_s)\,\mathrm{d}w_s\Big)\\ (Gx_k)(t) &= G\Big(t, x_t, \int_{t_k}^t g_1(t,s,x_s)\,\mathrm{d}s, \int_{t_k}^t g_2(t,s,x_s)\,\mathrm{d}w_s\Big)\\ (Fx_k^n)(t) &= F\Big(t, x_{t_k}^n, \int_{t_k}^t f_1(t,s,x_{t_k}^n)\,\mathrm{d}s, \int_{t_k}^t f_2(t,s,x_{t_k}^n)\,\mathrm{d}w_s\Big)\\ (Gx_k^n)(t) &= G\Big(t, x_{t_k}^n, \int_{t_k}^t g_1(t,s,x_{t_k}^n)\,\mathrm{d}s, \int_{t_k}^t g_2(t,s,x_{t_k}^n)\,\mathrm{d}w_s\Big)\end{aligned}$$

In connection with the introduced notations, the equations (1) and (5) can be expressed in the shorter integral forms,

$$x_t = x_0 + \int_0^t (Fx_0)(s)\,\mathrm{d}s + \int_0^t (Gx_0)(s)\,\mathrm{d}w_s,\quad t\in[0,1], \tag{6}$$

$$x_t^n = x_{t_k}^n + \int_{t_k}^t (Fx_k^n)(s)\,\mathrm{d}s + \int_{t_k}^t (Gx_k^n)(s)\,\mathrm{d}w_s,\quad t\in[t_k,t_{k+1}]. \tag{7}$$

First, let us prove some auxiliary results.

Proposition 1. *Let* $E|x_0|^p < \infty$ *for* $p \geq 2$*, the conditions* (2) *and* (3) *be satisfied and* x^n *be the solution of Eq.* (5)*. Then,*

$$E|x_t^n - x_{t_k}^n|^p \leq Q \cdot \delta_n^{p/2}, \quad t \in [t_k, t_{k+1}], \quad 0 \leq k \leq n-1,$$

where Q *is a generic constant independent of* n *and* k*.*

Proof. Let us note that by virtue of the earlier cited existence and uniqueness theorem, one can prove that $E\sup_{t\in[0,1]}|x_t^n|^p < M$ for any constant $M > 0$, independent of n and k.

In order to estimate $E|x_t^n - x_{t_k}^n|^p$, we shall first apply the elementary inequality $|a+b|^r \leq (2^{r-1} \vee 1)(\,|a|^r + |b|^r\,)$, $r \geq 0$, to Eq.(5) in integral form, Jensen's inequality and after that Hölder's inequality to Lebesgue's integral, as well as Burkholder-Davis-Gundy inequality [**4**], [**7**] to Ito's integral: For any $l > 0$, there exists a constant $c_l > 0$, such that $E \sup_{s\in[t_0,t]} \Big|\int_{t_0}^{s} f_u \,\mathrm{d}w_u\Big|^l \leq c_l\, E\Big(\int_{t_0}^{t} |f_u|^2 \,\mathrm{d}u\Big)^{l/2}$, for any measurable $\mathcal{F}_t$-adapted process $(f_t, t \in [0,T])$ such that $\int_{t_0}^{T} |f_t|^2 \,\mathrm{d}t < \infty$ a.s. In fact, in our case we use this inequality in which the left hand side is minorized by omitting supremum. Therefore, for all $t \in [t_k, t_{k+1}], 0 \leq k \leq n-1$, we obtain

$$(8) \qquad E|x_t^n - x_{t_k}^n|^p \leq 2^{p-1}\Big(E\Big|\int_{t_k}^{t} (Fx_k^n)(s)\,\mathrm{d}s\Big|^p + E\Big|\int_{t_k}^{t} (Gx_k^n)(s)\,\mathrm{d}w_s\Big|^p\Big)$$

$$\leq 2^{p-1}\Big((t-t_k)^{p-1}\int_{t_k}^{t} E\big|(Fx_k^n)(s)\big|^p \,\mathrm{d}s + c_p(t-t_k)^{p/2-1}\int_{t_k}^{t} E\big|(Gx_k^n)(s)\big|^p \,\mathrm{d}s\Big)$$

$$\equiv 2^{p-1}\big(\,(t-t_k)^{p-1} \cdot J_1(t) + c_p(t-t_k)^{p/2-1} \cdot J_2(t)\big),$$

To estimate $J_1(t)$, we shall apply the linear growth condition to the random functions F, f_1 and f_2 and the previously cited inequalities. Hence,

$$(9) \quad J_1(t) \leq L^p \int_{t_k}^{t} E\Big(1 + |x_{t_k}^n|^2 + \Big|\int_{t_k}^{s} f_1(s,r,x_{t_k}^n)\,\mathrm{d}r\Big|^2 + \Big|\int_{t_k}^{s} f_2(s,r,x_{t_k}^n) dw_r\Big|^2\Big)^{p/2} \mathrm{d}s$$

$$\leq 4^{p/2-1} L^p \int_{t_k}^{t} \Big(1 + E|x_{t_k}^n|^p + E\Big|\int_{t_k}^{s} f_1(s,r,x_{t_k}^n)\mathrm{d}r\Big|^p + E\Big|\int_{t_k}^{s} f_2(s,r,x_{t_k}^n) dw_r\Big|^p\Big)\mathrm{d}s$$

$$\leq 4^{p/2-1} L^p \int_{t_k}^{t} \Big[1 + E|x_{t_k}^n|^p + L^p\big((s-t_k)^{p-1} + c_p(s-t_k)^{p/2-1}\big)\int_{t_k}^{s} E(1 + |x_{t_k}^n|^2)^{p/2}\mathrm{d}r\Big]\,\mathrm{d}s$$

$$\leq 4^{p/2-1} L^p (1+M) \int_{t_k}^{t} \big[1 + 2^{p/2-1} L^p\big((s-t_k)^p + c_p(s-t_k)^{p/2}\big)\big]\,\mathrm{d}s$$

$$\leq C_1(L, M, c_p, p) \cdot (t - t_k),$$

where $C_1(L, M, c_p, p)$ is a generic constant. Similarly, by repeating completely the previous procedure, we find

$$J_2(t) \le C_2(L, M, c_p, p) \cdot (t - t_k), \tag{10}$$

where $C_2(L, M, c_p, p)$ is also a generic constant. Now, the relation (8) together with (9) and (10) implies that, for all $t \in [t_k, t_{k+1}],\ 0 \le k \le n-1$,

$$E|x_t^n - x_{t_k}^n|^p \le Q \cdot (t - t_k)^{p/2} \le Q \cdot \delta_n^{p/2},$$

where Q is a constant independent of n and k. □

Proposition 2. *Let $E|x_0|^p < \infty$ for $p \ge 2$, the conditions (2) and (3) be satisfied, and x and x^n be the solutions of the equations (6) and (7) respectively. Then,*

$$\sup_{t\in[0,1]} E|x_t - x_t^n|^p \le H \cdot \delta_n^{p/2},$$

where H is a generic constant independent of n and k.

Proof. Let $p > 2$ and $t \in [t_k, t_{k+1}]$. If we subtract Eq. (6) and (7) and after that apply Itô's differential formula to $|x_t - x_t^n|^p$, we get

$$E|x_t - x_t^n|^p \le E|x_{t_k} - x_{t_k}^n|^p + p \cdot I_1(t) + \frac{p(p-1)}{2} \cdot I_2(t) + p \cdot I_3(t),$$

where

$$I_1(t) = E\int_{t_k}^{t} \big((Fx_k)(s) - (Fx_k^n)(s) \big) |x_s - x_s^n|^{p-1} \, \mathrm{d}s$$

$$I_2(t) = E\int_{t_k}^{t} \big((Gx_k)(s) - (Gx_k^n)(s) \big)^2 |x_s - x_s^n|^{p-2} \, \mathrm{d}s$$

$$I_3(t) = E\int_{t_k}^{t} \big((Gx_k)(s) - (Gx_k^n)(s) \big) |x_s - x_s^n|^{p-1} \, \mathrm{d}w_s.$$

Let us denote that $\phi_t = E|x_t - x_t^n|^p$. Since $I_3(t) = 0$, we have

$$\phi_t = \phi_{t_k} + p \cdot I_1(t) + \frac{p(p-1)}{2} \cdot I_2(t), \quad t \in [t_k, t_{k+1}]. \tag{11}$$

First, let us estimate $I_1(t)$. Since F satisfies the LIPSCHITZ condition (2), it follows that

$$I_1(t) \le L\int_{t_k}^{t} \phi_s \, \mathrm{d}s + L\int_{t_k}^{t} E\Big(|x_s^n - x_{t_k}^n| + \Big| \int_{t_k}^{s} \big(f_1(s, r, x_r) - f_1(s, r, x_{t_k}^n) \big) \, \mathrm{d}r \Big| \tag{12}$$

$$+ \Big| \int_{t_k}^{s} \big(f_2(s, r, x_r) - f_2(s, r, x_{t_k}^n) \big) \, \mathrm{d}w_r \Big| \Big) |x_s - x_s^n|^{p-1} \mathrm{d}s.$$

By applying HÖLDER's inequality for $\nu = p, \mu = p/(p-1)$, to the second term in (12), after that the elementary inequality $|a|^r|b|^{1-r} \le r|a| + (1-r)|b|,\ 0 \le r \le 1$ and Proposition 1, we obtain

$$\int_{t_k}^{t} E|x_s^n - x_{t_k}^n|\,|x_s - x_s^n|^{p-1}\mathrm{d}s \le \int_{t_k}^{t} \big(E|x_s^n - x_{t_k}^n|^p\big)^{1/p}\phi_s^{(p-1)/p}\mathrm{d}s \tag{13}$$

$$\le \frac{1}{p}\,Q\,\delta_n^{p/2}\,(t-t_k) + \frac{p-1}{p}\int_{t_k}^{t}\phi_s\,\mathrm{d}s.$$

Since f_1 satisfies the LIPSCHITZ condition (2), it follows that

$$\int_{t_k}^{t} E\Big|\int_{t_k}^{s}\big(f_1(s,r,x_r) - f_1(s,r,x_{t_k}^n)\big)\,\mathrm{d}r\Big| \cdot |x_s - x_s^n|^{p-1}\mathrm{d}s \tag{14}$$

$$\le L\int_{t_k}^{t} E\int_{t_k}^{s}\big(|x_r - x_r^n| + |x_r^n - x_{t_k}^n|\big)\,dr \cdot |x_s - x_s^n|^{p-1}\mathrm{d}s$$

$$\le L\int_{t_k}^{t}\Bigg(\Big(E\Big|\int_{t_k}^{s}|x_r - x_r^n|\,\mathrm{d}r\Big|^p\Big)^{1/p} + \Big(E\Big|\int_{t_k}^{s}|x_r^n - x_{t_k}^n|\,dr\Big|^p\Big)^{1/p}\Bigg)\,\phi_s^{(p-1)/p}\,\mathrm{d}s$$

$$\le L\Big(\frac{1}{p}\int_{t_k}^{t}(s-t_k)^{p-1}\int_{t_k}^{s}\big(\phi_r + E|x_r^n - x_{t_k}^n|^p\big)\,\mathrm{d}r\,\mathrm{d}s + \frac{2(p-1)}{p}\int_{t_k}^{t}\phi_s\,\mathrm{d}s\Big)$$

$$\le L\Big(\frac{(t-t_k)^p}{p^2}\int_{t_k}^{t}\phi_s\,\mathrm{d}s + Q\,\delta_n^{p/2}\,\frac{(t-t_k)^{p+1}}{p(p+1)} + \frac{2(p-1)}{p}\int_{t_k}^{t}\phi_s\,\mathrm{d}s\Big).$$

Similarly,

$$\int_{t_k}^{t} E\Big|\int_{t_k}^{s}\big(f_2(s,r,x_r) - f_2(s,r,x_{t_k}^n)\big)\,\mathrm{d}w_r\Big| \cdot |x_s - x_s^n|^{p-1}\mathrm{d}s \tag{15}$$

$$\le \int_{t_k}^{t}\Bigg(\Big(E\Big|\int_{t_k}^{s}\big(f_2(s,r,x_r) - f_2(s,r,x_r^n)\big)\,\mathrm{d}w_r\Big|^p\Big)^{1/p}$$

$$+ \int_{t_k}^{t}\Big(E\Big|\int_{t_k}^{s}\big(f_2(s,r,x_r^n) - f_2(s,r,x_{t_k}^n)\big)\,\mathrm{d}w_r\Big|^p\Big)^{1/p}\Bigg)\,\phi_s^{(p-1)/p}\,\mathrm{d}s$$

$$\le L\Big(c_p\,\frac{(t-t_k)^{p/2}}{p^2}\int_{t_k}^{t}\phi_s\,\mathrm{d}s + Q\,\delta_n^{p/2}\,\frac{(t-t_k)^{p/2+1}}{p(p/2+1)} + \frac{2(p-1)}{p}\int_{t_k}^{t}\phi_s\,\mathrm{d}s\Big).$$

Now, by taking (13), (14) and (15) to (12), we deduce that, for $t \in [t_k, t_{k+1}]$,

$$I_1(t) \le \alpha_1(L,Q,c_p,p)\cdot\delta_n^{p/2}\,(t-t_k) + \beta_1(L,c_p,p)\int_{t_k}^{t}\phi_s\,\mathrm{d}s,$$

where α_1 and β_1 are generic constants.

Analogously, in order to estimate $I_2(t)$, we shall employ HÖLDER's inequality for $\nu = p/2, \mu = p/(p-2)$ and the above used procedure. Finally,

$$I_2(t) \le \alpha_2(L, Q, c_p, p) \cdot \delta_n^{p/2}\,(t - t_k) + \beta_2(L, c_p, p) \int_{t_k}^{t} \phi_s \,\mathrm{d}s.$$

So, from (11) and the estimated values $I_1(t)$ and $I_2(t)$ we conclude that

$$\phi_t \le \phi_{t_k} + \alpha\, \delta_n^{p/2}(t - t_k) + \beta \int_{t_k}^{t} \phi_s \,\mathrm{d}s, \quad t \in [t_k, t_{k+1}],\ 0 \le k \le n-1,$$

where $\alpha = \alpha(L, Q, c_p, p)$ and $\beta = \beta(L, Q, c_p, p)$ are generic constants. An application of the well-known GRONWALL-BELLMAN's inequality [**2**] yields

$$\phi_t \le \big(\,\phi_{t_k} + \alpha\, \delta_n^{p/2}(t - t_k)\,\big) \cdot e^{\beta(t - t_k)}, \quad t \in [t_k, t_{k+1}],\ 0 \le k \le n-1. \tag{16}$$

By taking $t = t_{k+1}$ in (16), we come to the following recurrence relation:

$$\phi_{t_{k+1}} \le \big[\,\phi_{t_k} + \alpha\, \delta_n^{p/2}(t_{k+1} - t_k)\,\big] \cdot e^{\beta(t_{k+1} - t_k)}, \quad 0 \le k \le n-1.$$

Since $\phi_{t_0} = E|x_0 - x_0^n|^p = 0$, we easily deduce

$$\phi_{t_k} \le \alpha\, \delta_n^{p/2} \sum_{i=0}^{k-1} (t_{i+1} - t_i)\, e^{\beta(t_k - t_i)} \le \alpha\, e^{\beta} \cdot \delta_n^{p/2}, \quad 0 \le k \le n-1. \tag{17}$$

Finally, from (16) it follows that ϕ_t is uniformly bounded on $[t_k, t_{k+1}]$, i.e. there exists a constant $H > 0$, that is

$$\phi_t \le H \cdot \delta_n^{p/2}, \quad t \in [t_k, t_{k+1}], \quad 0 \le k \le n-1,$$

and, therefore, $\sup\limits_{t \in [0,1]} \phi_t \le H \cdot \delta_n^{p/2}$.

The case $p = 2$ can be treated similarly. We estimate $I_1(t)$ by applying CAUCHY–SCHWARZ inequality, while the estimation for $I_2(t)$ goes directly in the sense of the L^2–stochastic integral isometry. Finally, we come again to the relation (16). Thus the proof becomes complete. □

In view of the preceding result, it is logical to expect that the sequence of the approximate solutions $\{x^n, n \in \mathbf{N}\}$ tends to the solution x as $\delta_n \to 0,\ n \to \infty$, in the L^p-norm. This assertion, as the main goal of the present paper, immediately follows from the next theorem, which gives an estimation of the speed of this convergence.

Theorem 1. *Let $E|x_0|^p < \infty$ for $p \ge 2$, the conditions* (2) *and* (3) *be satisfied, and x and x^n be the solutions of the equations* (6) *and* (7) *respectively. Then,*

$$E \sup_{t \in [0,1]} |x_t - x_t^n|^p = O\big(\delta_n^{p/2}\big), \quad \delta_n \to 0,\ n \to \infty.$$

Proof. To prove this assertion, without emphasizing any steps, we shall apply the previous treatment. Therefore,

$$\begin{aligned}
E\sup_{t\in[0,1]}|x_t-x_t^n|^p &\le 2^{p-1}\Big(E\sup_{t\in[0,1]}\Big|\int_0^t\big((Fx_0)(s)-(Fx_0^n)(s)\big)\,\mathrm{d}s\Big|^p\\
&\quad +E\sup_{t\in[0,1]}\Big|\int_0^t\big((Gx_0)(s)-(Gx_0^n)(s)\big)\,\mathrm{d}w_s\Big|^p\Big)\\
&\le 2^{p-1}\Big(E\Big|\int_0^1\big((Fx_0)(s)-(Fx_0^n)(s)\big)\,\mathrm{d}s\Big|^p\\
&\quad +c_p\,E\Big|\int_0^1\big|(Gx_0)(s)-(Gx_0^n)(s)\big|^2\,\mathrm{d}s\Big|^{p/2}\Big)\\
&\le 2^{p-1}\Big(\int_0^1 E\big|(Fx_0)(s)-(Fx_0^n)(s)\big|^p\mathrm{d}s\\
&\quad +c_p\int_0^1 E\big|(Gx_0)(s)-(Gx_0^n)(s)\big|^p\mathrm{d}s\Big)\\
&\le 6^{p-1}L^p(1+c_p)\int_0^1 E\Big(|x_s-x_s^n|^p+\Big|\int_0^s\big(f_1(s,r,x_r)-f_1(s,r,x_r^n)\big)\,\mathrm{d}r\Big|^p\\
&\quad +\Big|\int_0^s\big(f_2(s,r,x_r)-f_2(s,r,x_r^n)\big)\,dw_r\Big|^p\Big)\,\mathrm{d}s\\
&\le 6^{p-1}L^p(1+c_p)\int_0^1\Big(E|x_s-x_s^n|^p\\
&\quad +L^p\big(s^{p-1}+c_ps^{p/2-1}\big)\int_0^s E|x_r-x_r^n|^p\,dr\Big)\,\mathrm{d}s.
\end{aligned}$$

This, in view of Proposition 2, yields

$$\begin{aligned}
E\sup_{t\in[0,1]}|x_t-x_t^n|^p &\le 6^{p-1}L^p(1+c_p)\Big(1+L^p\Big(\frac{1}{p+1}+\frac{c_p}{p/2+1}\Big)\Big)\cdot H\,\delta_n^{p/2}\\
&= O\big(\delta_n^{p/2}\big),\quad \delta_n\to 0,\ n\to\infty,
\end{aligned}$$

which establishes the theorem. □

Therefore, $x^n \xrightarrow{L^p} x$ as $\delta_n\to 0,\ n\to\infty$.

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