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SOME PROPERTIES OF TWO LINEAR OPERATORS

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In this paper we study - by means of the "umbral calculus" (see [**3**], [**4**], [**5**]) and the TCHEBICHEFF polynomials - some properties of the linear operators $P_t = E^t \sin(\sqrt{1-t^2}\,D)$, $Q_t = E^t \cos(\sqrt{1-t^2}\,D)$, where E^t is the shift-operator and D-the derivative, for example the relation between ABEL operator $A = DE^a$ and the operators P_t and Q_t. The author thanks prof. dr. ALEXANDRU LUPAŞ for his generous suggestions.

1. INTRODUCTION

Let us denote by Π the (complex) linear space of all polynomials with real coefficients. Let us put in evidence some operators $\Pi \to \Pi$. For instance, I is the identity, D-the derivative, E^a is the shift-operator $(E^a f)(x) = f(x+a)$.

It is known that (see [**3**])

$$\big(e^{(t+i\sqrt{1-t^2}\,)D} f\big)(x_0) = f(x_0 + t + i\sqrt{1-t^2}\,), \tag{1.1}$$

where $f \in \Pi$, x_0, $t \in \mathbf{R}$, $|t| < 1$.

We have

$$\begin{aligned} f(x_0 + t + i\sqrt{1-t^2}\,) &- f(x_0 + t - i\sqrt{1-t^2}\,) \\ &= \big(e^{tD}(\cos\sqrt{1-x^2}\,D + i\sin\sqrt{1-t^2}\,D)f\big)(x_0) \\ &\quad -\big(e^{tD}(\cos\sqrt{1-x^2}\,D - i\sin\sqrt{1-t^2}\,D)f\big)(x_0) \\ &= 2i\,\big((e^{tD}\sin\sqrt{1-t^2}\,D)f\big)(x_0) \end{aligned} \tag{1.2}$$

and

$$f(x_0+t+i\sqrt{1-t^2}\,) + f(x_0+t-i\sqrt{1-t^2}\,) = 2\big((e^{tD}\cos\sqrt{1-t^2}\,D)f\big)(x_0). \tag{1.3}$$

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Now, it is natural to consider the delta-operator $P_t = E^t \sin(\sqrt{1-t^2}\, D)$ and the linear operator $Q_t = E^t \cos(\sqrt{1-t^2}\, D)$.

In the following we study some properties of the linear operators P_t and Q_t.

2. THE DELTA–OPERATOR P_t

We consider the TAYLOR expansion $f(x) = \sum\limits_{k\geq 0} a_k (x-x_0)^k$ with $a_k = \dfrac{f^{(k)}(x_0)}{k!}$.

We observe that

$$\frac{f(x_0+t+i\sqrt{1-t^2}\,) - f(x_0+t-i\sqrt{1-t^2}\,)}{2} = \sum_{k\geq 1} a_k \frac{(t+i\sqrt{1-t^2}\,)^k - (t-i\sqrt{1-t^2}\,)^k}{2}$$

and noting $\widetilde{\varphi} = \arccos t$, $\widetilde{\varphi} \in (0,\pi)$, we obtain

$$\frac{f(x_0+t+i\sqrt{1-t^2}\,) - f(x_0+t-i\sqrt{1-t^2}\,)}{2} = i \sum_{k\geq 1} a_k \sin k\widetilde{\varphi}.$$

Further, let U_k denotes TCHEBICHEFF polynomials of the second kind

$$U_k = \frac{\sin\big((k+1)\arccos t\big)}{(k+1)\sin(\arccos t)}\,, \quad k\in \mathbf{N}, \quad |t|<1.$$

We have

$$\begin{aligned}\frac{f(x_0+t+i\sqrt{1-t^2}\,) - (x_0+t-i\sqrt{1-t^2}\,)}{2} &= i\sum_{k\geq 1} k a_k (\sin\widetilde{\varphi}) U_{k-1}(t)\\ &= i\sqrt{1-t^2}\sum_{k\geq 0}(k+1)a_{k+1}U_k(t).\end{aligned}$$

Taking account that $\int\limits_{-1}^{1} U_k(t)U_j(t)\sqrt{1-t^2}\,\mathrm{d}t = 0$ for $k\neq j$, we get

$$\int_{-1}^{1} \frac{f(x_0+t+i\sqrt{1-t^2}\,) - f(x_0+t-i\sqrt{1-t^2}\,)}{2}\, U_j(t)\,\mathrm{d}t = i\,(j+1)a_{j+1}\,\frac{1}{\delta_j}\,,$$

where

$$\delta_j^{-1} = \int_{-1}^{1} U_j^{\,2}(t)\sqrt{1-t^2}\,\mathrm{d}t = \frac{\pi}{2\,(j+1)^2}\,.$$

Hence

$$(j+1)\,a_{j+1} = \frac{1}{i}\int_{-1}^{1} \frac{f(x_0+t+i\sqrt{1-t^2}\,) - f(x_0+t-i\sqrt{1-t^2}\,)}{2}\,\delta_j U_j(t)\,\mathrm{d}t.$$

Now, according to $f'(x) = \sum\limits_{k\geq 0}(k+1)a_{k+1}(x-x_0)^k$ we have

$$f'(x) = \frac{1}{i}\int_{-1}^{1} \frac{f(x_0+t+i\sqrt{1-t^2}\,) - f(x_0+t-i\sqrt{1-t^2}\,)}{2}\Big(\sum_{k\geq 0}\delta_k(x-x_0)^k U_k(t)\Big)\,\mathrm{d}t.$$

Using next the generating relation

$$\frac{2}{\pi}\,\frac{1-x^2}{(1-2tx+x^2)^2} = \sum_{k=0}^{\infty}\delta_k U_k(t)x^k \qquad (|x|<1,\ |t|<1)$$

we obtain

$$f'(t) = \frac{2}{i\pi}\int_{-1}^{1} \frac{f(x_0+t+i\sqrt{1-t^2}\,) - f(x_0+t-i\sqrt{1-t^2}\,)}{2} \cdot \frac{1-(x-x_0)^2}{\big(1-2t\,(x-x_0)+(x-x_0)^2\big)^2}\,\mathrm{d}t.$$

Hence

$$f'(x_0+z) = \frac{2}{i\pi}\int_{-1}^{1} \frac{f(x_0+t+i\sqrt{1-t^2}\,) - f(x_0+t-i\sqrt{1-t^2}\,)}{2} \cdot \frac{1-z^2}{(1-2tz+z^2)^2}\,\mathrm{d}t. \tag{2.1}$$

Further, it is know that the Abel operator $A = DE^a$ is a delta-operator and the basic set for this operator $p_n^{(a)} = x(x-na)^{n-1}$, (see [**4**]). Regarding the relation between the operators A and P_t, we have

Theorem 1. *Suppose that* $|a|<1$, *and* $f\in\Pi$. *Then*

$$(Af)(x) = \frac{2(1-a^2)}{\pi}\int_{-1}^{1}(P_t f)(x)\,\frac{\mathrm{d}t}{(1-2ta+a^2)^2}\,, \textit{ so} \tag{2.2}$$

$$A = \frac{2(1-a^2)}{\pi}\int_{-1}^{1}\frac{1}{(1-2ta+a^2)^2}\,P_t\,\mathrm{d}t. \tag{2.3}$$

Proof. It is easy to observe that (2.2) follows from (1.2) and (2.1), with $z = a$ and $x_0 = x$.

Let us observe that for the Abel polynomials we have

$$p_n^{(a)}(x) = \frac{2(1-a^2)}{(n+1)\pi} \int_{-1}^{1} (P_t p_{n+1}^{(a)})(x) \frac{\mathrm{d}t}{(1-2ta+a^2)^2}\,, \quad |a|<1. \tag{2.4}$$

Corollary. *We have*

$$(Df)(x) = \frac{2}{\pi} \int_{-1}^{1} (P_t f)(x)\,\mathrm{d}t, \quad \textit{so} \tag{2.5}$$

$$D = \frac{2}{\pi} \int_{-1}^{1} P_t\,\mathrm{d}t. \tag{2.6}$$

Because P_t is a delta-operator we will try to find an expression for the inverse of this Pincherle derivative.

Theorem 2. *If* $t = \cos\widetilde{\varphi}$, $\widetilde{\varphi} \in (0,\pi)$, *then* $P_t'^{\,-1} = e^{-tD}\mathrm{cosec}\,\big(\widetilde{\varphi} I + (\sin\widetilde{\varphi})D\big)$.

Proof. We consider $h(t,z) = e^{tz}\sin(\sqrt{1-t^2}\,z)$ whence $h_z'(t,z) = e^{tz}\sin\big(\widetilde{\varphi} + (\sin\widetilde{\varphi})z\big)$. We find $P_t'^{\,-1} = e^{-tD}\mathrm{cosec}\,\big(\widetilde{\varphi} I + (\sin\widetilde{\varphi})D\big)$.

3. THE LINEAR OPERATOR Q_t

It is not difficult to obtain

$$\begin{aligned} &\frac{f(x_0+t+i\sqrt{1-t^2}\,) + f(x_0+t-i\sqrt{1-t^2}\,)}{2} \\ &\qquad = \sum_{k\ge 0} a_k \frac{(t+i\sqrt{1-t^2}\,)^k + (t-i\sqrt{1-t^2}\,)^k}{2} = \sum_{k\ge 0} a_k T_k(t), \end{aligned} \tag{3.1}$$

where $T_n(t) = \cos(n \arccos t)$, $n \in \mathbf{N}$, $|t|<1$, is the Tchebicheff polynomials of the first kind.

Because

$$\int_{-1}^{1} \frac{T_n(t)T_m(t)}{\sqrt{1-t^2}}\,\mathrm{d}t = \begin{cases} 0 & (m \ne n), \\ \pi/2 & (m = n \ne 0), \\ \pi & (m = n = 0), \end{cases}$$

we get from (3.1)

$$\int_{-1}^{1} \frac{f(x_0+t+i\sqrt{1-t^2}\,) + f(x_0+t-i\sqrt{1-t^2}\,)}{2}\, T_j(t) \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \frac{1}{\gamma_j}\, a_j,$$

where $\gamma_j = 1/\pi$ $(j \neq 0)$ and $\gamma_j = 2/\pi$ $(j \geq 1)$.

We have

$$f(x) = \int_{-1}^{1} \frac{f(x_0 + t + i\sqrt{1-t^2}\,) + f(x_0 + t - i\sqrt{1-t^2}\,)}{2} \Big(\sum_{k \geq 0} \gamma_k (x - x_0)^k T_k(t) \Big) \frac{\mathrm{d}t}{\sqrt{1-t^2}}$$

and using the generating relation

$$\frac{1}{\pi} \frac{1 - x^2}{1 - 2tx + x^2} = \sum_{k=0}^{\infty} \gamma_k T_k(t) x^k, \quad |x| < 1, \quad |t| < 1$$

we obtain

$$f(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{f(x_0 + t + i\sqrt{1-t^2}\,) + f(x_0 + t - i\sqrt{1-t^2}\,)}{2} \cdot \frac{1 - (x - x_0)^2}{1 - 2t(x - x_0) + (x - x_0)^2} \cdot \frac{\mathrm{d}t}{\sqrt{1-t^2}}\,. \tag{3.2}$$

Hence

$$f(x_0 + z) = \frac{1 - z^2}{\pi} \int_{-1}^{1} \frac{f(x_0 + t + i\sqrt{1-t^2}\,) + f(x_0 + t - i\sqrt{1-t^2}\,)}{2\,(1 - 2tz + z^2)} \cdot \frac{\mathrm{d}t}{\sqrt{1-t^2}}\,. \tag{3.3}$$

Theorem 3. *For $|a| < 1$ and $f \in \Pi$, we have*

$$(E^a f)(x) = \frac{1 - a^2}{\pi} \int_{-1}^{1} (Q_t f)(x) \, \frac{\mathrm{d}t}{(1 - 2ta + a^2)\sqrt{1-t^2}}\,, \quad \textit{so} \tag{3.4}$$

$$E^a = \frac{1 - a^2}{\pi} \int_{-1}^{1} \frac{1}{(1 - 2ta + a^2)\sqrt{1-t^2}} \, Q_t \, \mathrm{d}t. \tag{3.5}$$

Proof. Immediate from (1.3) and (3.3).

Corollary. *If $f \in \Pi$, then*

$$f(x) = \frac{1}{\pi} \int_{-1}^{1} (Q_t f)(x) \, \frac{\mathrm{d}t}{\sqrt{1-t^2}}\,, \quad \textit{so} \tag{3.6}$$

$$I = \frac{1}{\pi} \int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} \, Q_t \, \mathrm{d}t. \tag{3.7}$$

Since Q_t is an invertible shift-invariant operator, we can consider the delta-operator $R_t = DQ_t$.

Regarding the relation between the ABEL operator $A = DE^a$ and R_t we have

Theorem 4. *For* $|a| < 1$,

$$(Af)(x) = \frac{1-a^2}{\pi} \int_{-1}^{1} (R_t f)(x) \frac{\mathrm{d}t}{(1-2ta+a^2)\sqrt{1-t^2}}\,. \tag{3.8}$$

Proof. Immediate from Theorem 3.

Theorem 5. *For* $f \in \Pi$, *we have*

$$\int_{-1}^{1} (R_t f)(x) \frac{\mathrm{d}t}{\sqrt{1-t^2}} = 2 \int_{-1}^{1} (P_t f)(x)\,\mathrm{d}t. \tag{3.9}$$

Proof. Immediate from (2.5) and (3.6).

Theorem 6. *If* $t = \cos\widetilde{\varphi}$, $\widetilde{\varphi} \in (0,\pi)$, *then* $R_t'^{-1} = g(D)$, *where*

$$g(z) = \frac{e^{-tz}}{\cos(\sqrt{1-t^2}\,z) + z\cos(\widetilde{\varphi} + \sqrt{1-t^2}\,z)}\,.$$

Proof. We consider $h(t,z) = ze^{tz}\cos(\sqrt{1-t^2}\,z)$ and hence

$$h_z'(t,z) = e^{tz}\big(\cos(\sqrt{1-t^2}\,z) + z\cos(\widetilde{\varphi} + \sqrt{1-t^2}\,z)\big).$$

We note with $g(z)$ the formal series

$$g(z) = \frac{e^{-tz}}{\cos(\sqrt{1-t^2}\,z) + z\cos(\widetilde{\varphi} + \sqrt{1-t^2}\,z)}$$

and we have $R_t'^{-1} = g(D)$.

Theorem 7. *For any* $f \in \Pi$, *we have*

$$\int_{-1}^{1} P_t\Big(\int_{-1}^{1} (Q_t f)(x) \frac{\mathrm{d}t}{\sqrt{1-t^2}}\Big)(x)\,\mathrm{d}t = \int_{-1}^{1} Q_t\Big(\int_{-1}^{1} (P_t f)(x)\,\mathrm{d}t\Big)(x) \frac{\mathrm{d}t}{\sqrt{1-t^2}}\,. \tag{3.10}$$

Proof. Using (2.5) and (3.6) we obtain

$$\int_{-1}^{1} P_t\Big(\int_{-1}^{1} (Q_t f)(x) \frac{\mathrm{d}t}{\sqrt{1-t^2}}\Big)(x)\,\mathrm{d}t = \pi \int_{-1}^{1} (P_t f)(x)\,\mathrm{d}t = \frac{\pi^2}{2} f'(x)$$

$$= \frac{\pi}{2} \int_{-1}^{1} (Q_t f')(x) \frac{\mathrm{d}t}{\sqrt{1-t^2}} = \int_{-1}^{1} Q_t\Big(\int_{-1}^{1} P_t f(x)\,\mathrm{d}t\Big)(x) \frac{\mathrm{d}t}{\sqrt{1-t^2}} \qquad \text{Q.E.D.}$$

Regarding the operator differential equation satisfied by the linear operators P_t and Q_t we have

Theorem 8. *The delta-operator P_t and the linear operator Q_t satisfies the operator differential equation in the Pincherle derivative $Y'' - 2tY' + Y = 0$.*

Proof. We have

$$P_t' = tP_t + \sqrt{1-t^2}\,Q_t, \qquad Q_t' = -\sqrt{1-t^2}\,P_t + tQ_t$$

and

$$\begin{aligned} P_t'' &= tP_t' + \sqrt{1-t^2}\,Q_t' = (2t^2-1)P_t + 2t\sqrt{1-t^2}\,Q_t, \\ Q_t'' &= -\sqrt{1-t^2}\,P_t' + tQ_t' = -2t\sqrt{1-t^2}\,P_t + (2t^2-1)\,Q_t. \end{aligned}$$

Whence we obtain $-2tP_t' + P_t'' = -P_t, \quad -2tQ_t' + Q_t'' = -Q_t$.
Hence $P_t'' - 2tP_t + P_t = 0, \quad Q_t'' - 2tQ_t + Q_t = 0$.

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