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PRESEMINORM GENERATING RELATIONS AND THEIR MINKOWSKI FUNCTIONALS

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We show that instead of the Minkowski functionals of summative sequences of absorbing, balanced subsets of a vector space X it is more convenient to consider first the Minkowski functionals of absorbing, balanced valued additive relations of certain dense subsets of $\mathbf{R}_+$ onto X.

0. INTRODUCTION

In this paper, motivated by the results of [**5**] and the metrization theorems of [**2**, p. 18] and [**1**, p. 11], we shall introduce and investigate the following definitions.

A dense subset D of the set $\mathbf{R}_+$ of all positive numbers is called admissible if

(1) $r+s \in D$ for all $r,\ s \in D$;

(2) $r-s \in D$ for all $r,\ sD$ with $s<r$.

A relation F of an admissible subset D_F of $\mathbf{R}_+$ onto a vector space X is called a preseminorm generating relation for X if

(1) $F(r)+F(s) \subset F(r+s)$ for all $r,\ s \in D_F$;

(2) $F(r)$ is an absorbing, balanced subset of X for all $r \in D_F$.

If p is a preseminorm on X, then the relations F_p and $\overline{F}_p$ on $\mathbf{R}_+$ to X, defined by

$$F_p(r) = p^{-1}\big(\,[0,\ r[\,\big) \quad \text{and} \quad \overline{F}_p(r) = p^{-1}\big(\,[0,\ r]\,\big)$$

for all $r \in \mathbf{R}_+$, are called the lower and upper preseminorm generating relations induced by p.

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If F is a preseminorm generating relation for X, then the function p_F, defined by $p_F(x) = \inf\big(F^{-1}(x)\big)$ for all $x \in X$, is called the MINKOWSKI functional or gauge of F.

The set $N_F = \bigcap_{r \in D_F} F(r)$ is called the kernel of F.

The relations F_* and F^* on $\mathbf{R}_+$ to X, defined by

$$F_*(r) = \bigcup_{u<r} F(u) \quad \text{and} \quad F^*(r) = \bigcap_{r<v\in D_F} F(v)$$

for all $r \in \mathbf{R}_+$, are called the lower and upper regularizations of F.

Moreover, the relation $\widetilde{F} = F \cup F_*$ is called the natural extension of F.

To let the reader feel the appropriateness of the above definitions, we shall only quote here the following three theorems.

Theorem 1. *If p is a preseminorm on X, then F_p and $\overline{F}_p$ are preseminorm generating relations for X such that $F_p \subset \overline{F}_p$. Moreover,*

$$N_{F_p} = N_{\overline{F}_p} = p^{-1}(0), \quad F_p = \big(F_p\big)_* = \big(\overline{F}_p\big)_*, \quad \overline{F}_p = \big(F_p\big)^* = \big(\overline{F}_p\big)^*.$$

REMARK 1. Therefore, the equalities $F_p = \big(F_p\big)^*$, $\overline{F}_p = \big(\overline{F}_p\big)_*$, $F_p = \overline{F}_p$, $p = 0$, $F_p = \mathbf{R}_+ \times X$ and $\overline{F}_p = \mathbf{R}_+ \times X$ are equivalent.

Theorem 2. *If F is a preseminorm generating relation for X, then p_F is the unique preseminorm on X such that*

$$F_{p_F}(r) \subset F(r) \subset \overline{F}_{p_F}(r)$$

for all $r \in D_F$. Moreover, $N_F = p_F^{-1}(0)$, and $F_* = F_{p_F}$ and $F^* = \overline{F}_{p_F}$.

REMARK 2. Therefore, $p_F = p_{F_*} = p_{F^*}$ and $N_F = N_{F_*} = N_{F^*}$, and moreover $F_* = \big(F_*\big)_* = \big(F^*\big)_*$ and $F^* = \big(F_*\big)^* = \big(F^*\big)^*$.

Theorem 3. *If F is a preseminorm generating relation for X, then $\widetilde{F}$ is a preseminorm generating relation for X such that*

$$\widetilde{F}(r) = F(r) \ \text{ if } r \in D_F \ \text{ and } \ \widetilde{F}(r) = F_*(r) \ \text{ if } r \in \mathbf{R}_+ \setminus D_F.$$

Moreover, $p_F = p_{\widetilde{F}}$ and $N_F = N_{\widetilde{F}}$, and $F_* = \big(\widetilde{F}\big)_*$ and $F^* = \big(\widetilde{F}\big)^*$.

REMARK 3. Unfortunately, if F is homogeneous in the sense that

$$rF(s) \subset F(rs)$$

for all r, $s \in D_F$, then in contrast to the regularizations F_* and F^*, the extension $\widetilde{F}$ need not be homogeneous.

1. PREREQUISITES

A subset F of a product set $X \times Y$ is called a relation on X to Y. If in particular $X = Y$, then we simply say that F is a relation on X. Note that if F is a relation on X to Y, then F is also a relation on $X \cup Y$.

If F is a relation on X to Y, and moreover $x \in X$ and $A \subset X$, then the sets $F(x) = \{y \in X : (x, y) \in F\}$ and $F[A] = \bigcup_{x\in A} F(x)$ are called the images of x and A under F. If $A \notin X$, then we may write $F(A)$ in place of $F[A]$.

If F is a relation on X to Y, then the sets $D_F = \{x \in X : F(x) \neq \emptyset\}$ and $R_F = F[D_F]$ are called the domain and range of F, respectively. If in particular $X = D_F$ (and $Y = R_F$), then we say that F is a relation of X into (onto) Y.

A relation F on X to Y is said to be a function if for each $x \in D_F$ there exists a unique $y \in Y$ such that $y \in F(x)$. In this case, by identifying singletons with their elements, we usually write $F(x) = y$ in place of $F(x) = \{y\}$.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x\in X}\{x\} \times F(x)$. Therefore, the inverse relation F^{-1} of F can be defined such that $F^{-1}(x) = \{y \in Y : x \in F(y)\}$ for all $x \in X$.

Throughout in the sequel, X will denote a vector space over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. And for any $\lambda \in \mathbf{K}$ and $A, B \subset X$ we write $\lambda A = \{\lambda x : x \in A\}$ and $A + B = \{x + y : x \in A, y \in B\}$.

Note that thus two axioms of a vector space may fail to hold for the family $\mathcal{P}(X)$ of all subsets of X. Namely, only the one-point subsets of X can have additive inverses. Moreover, in general, we only have $(\lambda + \mu) A \subset \lambda A + \mu A$.

If $A \subset X$ and $D \subset \mathbf{R}_+$, then we say that:

(1) A is D–absorbing if $X = \bigcup_{r\in D} rA$;

(2) A is balanced if $\lambda A \subset A$ for all $\lambda \in \mathbf{K}$ with $|\lambda| \leq 1$;

(3) A is D–convex if $rA + (1 - r)A \subset A$ for all $r \in D$ with $r < 1$.

In particular, the set A is called absorbing (convex) if it is $\mathbf{R}_+$–absorbing ($\mathbf{R}_+$–convex).

A function p of X into $\mathbf{R}$ is called a preseminorm [**3**] on X if

(1) $\lim_{\lambda\to 0} p(\lambda x) = 0$ for all $x \in X$;

(2) $p(\lambda x) \leq p(x)$ for all $\lambda \in \mathbf{K}$, with $|\lambda| \leq 1$, and $x \in X$;

(3) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in X$.

In particular, a preseminorm p on X is called a seminorm if instead of the conditions (1) and (2) the condition $p(\lambda x) \leq |\lambda| p(x)$ holds for all $\lambda \in \mathbf{K}$ and $x \in X$. Moreover, a seminorm (preseminorm) p is called a norm (prenorm) if $p(x) = 0$ implies $x = 0$.

If p is a preseminorm on X, then for each $r \in \mathbf{R}_+$ the relations B^p_r and $\bar{B}^p_r$ on X, defined by

$$B^p_r(x) = \big\{y \in X : p\,(x - y) < r\big\} \quad\text{and}\quad \bar{B}^p_r(x) = \big\{y \in X : p(x - y) \leq r\big\}$$

for all $x \in X$, are called the r-sized open and closed p–surroundings in X, respectively.

Concerning the above basic concepts we shall only need here the following simple theorems.

Theorem 1.1. *If $A \subset X$ and $D \subset \mathbf{R}_+$ such that D is closed under addition and division, then the following assertions hold*:

(1) *if A is D–convex, then $(r+s)A = rA + sA$ for all $r,\ s \in D$;*

(2) *if A is balanced, then $\lambda A \subset \mu A$ for all $\lambda,\ \mu \in \mathbf{K}$ with $|\lambda| \leq |\mu|$.*

Remark 1.2. Therefore, a balanced subset A of X is absorbing if and only if it is $\mathbf{N}$–absorbing.

Theorem 1.3. *If p is a preseminorm on X and $x \in X$, then*

(1) $p(0) = 0$; (2) $p(x) \geq 0$;

(3) $\big|\, p(x) - p(y) \,\big| \leq p(x-y)$ *for all* $x,\ y \in X$;

(4) $p(\lambda x) \leq p(\mu x)$ *for all* $\lambda,\ \mu \in \mathbf{K}$ *with* $|\lambda| \leq |\mu|$;

(5) $p(\lambda x) \leq np(x)$ *for all* $\lambda \in \mathbf{K}$ *and* $n \in \mathbf{N}$ *with* $|\lambda| \leq n$;

(6) $|m|^{-1} p(nx) \leq p(nm^{-1}x) \leq |n|\, p(m^{-1}x)$ *for all* $n,\ m \in \mathbf{Z}$ *with* $m \neq 0$.

Remark 1.4. If in particular p is a seminorm, then we can also state that $p(\lambda x) = |\lambda|\, p(x)$ for all $\lambda \in \mathbf{K}$ and $x \in X$. Therefore, our present definition of a seminorm coincides with the usual one.

Theorem 1.5. *If p is a preseminorm on X and $r,\ s \in \mathbf{R}_+$, then*

(1) $B_r^p(x) = x + B_r^p(0)$ *for all* $x \in X$;

(2) $B_r^p(0)$ *is an absorbing, balanced subset of X such that*

$$B_r^p(0) + B_s^p(0) \subset B_{r+s}^p(0).$$

Remark 1.6. If in particular p is a seminorm, then we can also state that $B_r^p(0)$ is convex and $B_r^p(0) = rB_1^p(0)$. Moreover, the same statements hold for the closed surroundings $\bar{B}_r^p$.

2. PRESEMINORM GENERATING RELATIONS AND THEIR BASIC PROPERTIES

Definition 2.1. *An order dense subset D of $\mathbf{R}+$ will be called admissible if*

(1) $r+s \in D$ *for all* $r,\ s \in D$;

(2) $r-s \in D$ *for all* $r,\ s \in D$ *with* $s < r$.

Remark 2.2. An admissible set D will be called regular if $rs \in D$ for all $r,\ s \in D$. Moreover, a regular admissible set D will be called normal if $r^{-1} \in D$ for all $r \in D$.

Clearly, $\mathbf{R}_+$ is a normal admissible subset of itself. Moreover, to provide a proper example for normal admissible sets, we can at once state

Example 2.3. The set $\mathbf{Q}_+$ of all positive rational numbers is the smallest normal admissible subset of $\mathbf{R}_+$.

Remark 2.4. In [**6**], it is shown that the set $\mathbf{D}_+$ of all positive dyadic rational numbers is a regular, but not normal admissible subset of $\mathbf{R}_+$.

Definition 2.5. *A relation F of an admissible subset D_F of $\mathbf{R}_+$ onto X will be called a preseminorm generating relation for X if*

(1) $F(r)+F(s)\subset F(r+s)$ *for all* $r,\ s\in D_F$;

(2) $F(r)$ *is an absorbing, balanced subset of* X *for all* $r\in D_F$.

Remark 2.6. A preseminorm generating relation F will be called homogeneous if D_F is regular and $r\,F(s)\subset F(rs)$ for all $r,\ s\in D_F$. Moreover, a preseminorm generating relation F will be called total if $D_F=\mathbf{R}_+$.

Note that the seminorm generating relations of [**5**] are total homogeneous preseminorm generating relations. Moreover, to provide some examples for not necessarily homogeneous preseminorm generating relations, we can at once state

Example 2.7. If p is a preseminorm on X and F_p and $\overline{F}_p$ are relations on $\mathbf{R}_+$ to X such that

$$F_p(r)=B_r^p(0)\quad\text{and}\quad \overline{F}_p(r)=\bar{B}_r^p(0)$$

for all $r\in\mathbf{R}_+$, then F_p and $\overline{F}_p$ are total preseminorm generating relations for X such that $F_p\subset\overline{F}_p$.

Remark 2.8. In [**6**], it is shown that a sequence $A=\big(A_n\big)_{n=0}^{\infty}$ of absorbing, balanced subsets of X naturally gives rise to a preseminorm generating relation F_A of $\mathbf{D}_+$ onto X whenever $A_{n+1}+A_{n+1}\subset A_n$ for all $n\in\mathbf{N}\cup\{0\}$.

The basic properties of preseminorm generating relations can be summarized in the following

Theorem 2.9. *If F is a preseminorm generating relation for X, then*

(1) $0\in F(r)$ *for all* $r\in D_F$;

(2) $F(r)\subset F(s)$ *for all* $r,\ s\in D_F$ *with* $r\le s$.

(3) $\lambda F(r)\subset F(nr)$ *for all* $\lambda\in\mathbf{K}$, $r\in D_F$ *and* $n\in\mathbf{N}$ *with* $|\lambda|\le n$.

Proof. The assertion (1) is already immediate from the fact that $F(r)$ is an absorbing subset of X for all $r\in D_F$.

While, to prove the assertions (2) and (3), it is enough to note only that

$$F(r)=F(r)+\{0\}\subset F(r)+F(s-r)\subset F(s)$$

for all $r,\ s\in D$ with $r<s$, and

$$\lambda F(r)\subset nF(r)\subset\sum_{k=1}^{n}F(r)\subset F(nr)$$

for all $\lambda \in \mathbf{K}$, $r \in D_F$ and $n \in \mathbf{N}$ with $|\lambda| \leq n$.

In addition to the assertion (3) of Theorem 2.9, it is also worth proving

Theorem 2.10. *If F is a preseminorm generating relation for X, then*

$$nF\big(m^{-1} r\big) \subset F\big(nm^{-1} r\big) \subset m^{-1}F\big(nr\big)$$

for all n, $m \in \mathbf{N}$ and $r \in \mathbf{R}_+$.

Proof. If $n \in \mathbf{N}$ and $r \in \mathbf{R}_+$, then by Theorem 2.9 (3) we have

$$nF(r) \subset F(nr),$$

since $F(r) = \emptyset$ whenever $r \notin D_F$. Hence, by writing $n^{-1} r$ in place of r, we get $nF(n^{-1} r) \subset F(r)$. Therefore, we also have

$$F(n^{-1} r) \subset n^{-1}F(r).$$

And hence, it is clear that the required inclusions are also true.

Now, as an immediate consequence of Theorem 2.10, we can also state

Corollary 2.11. *If F is a preseminorm generating relation for X, then*

$$F(r) = \bigcup_{n=1}^{\infty} nF\big(n^{-1}r\big) \quad \textit{and} \quad F(r) = \bigcap_{n=1}^{\infty} n^{-1}F\big(nr\big)$$

for all $r \in \mathbf{R}_+$.

Proof. By Theorem 2.10, we have

$$nF\big(n^{-1}r\big) \subset F(r) \subset n^{-1}F(nr)$$

for all $n \in \mathbf{N}$ and $r \in \mathbf{R}_+$. And hence, the required equalities are quite obvious.

3. THE MINKOWSKI FUNCTIONALS OF PRESEMINORM GENERATING RELATIONS

Definition 3.1. *If F is a preseminorm generating relation for X, then the function p_F defined by*

$$p_F(x) = \inf\,\big(F^{-1}(x)\big)$$

for all $x \in X$, will be called the Minkowski functional or gauge of F.

The appropriateness of the above definition is apparent from the following

Theorem 3.2. *If F is a preseminorm generating relation for X, then p_F is a preseminorm on X such that*

$$F_{p_F}(r) \subset F(r) \subset \overline{F}_{p_F}(r)$$

for all $r \in D_F$.

Proof. Assume that $x \in X$ and $\varepsilon > 0$. Then, since D_F is dense in $\mathbf{R}_+$, there exists an $r \in D_F$ such that $r < \varepsilon$. Moreover, since $F(r)$ is absorbing in X, there exists a $t \in \mathbf{R}_+$ such that $x \in t\,F(r)$. Define $\delta = t^{-1}$, and assume that $\lambda \in \mathbf{K}$ such that $|\lambda| < \delta$. Then, since $|\lambda t| = |\lambda| t < \delta t = 1$ and $F(r)$ is balanced, we have

$$\lambda x \in \lambda t F(r) \subset F(r).$$

Hence, it follows that $r \in F^{-1}(\lambda x)$, and thus

$$0 \leq p_F(\lambda x) = \inf \big(F^{-1}(\lambda x) \big) \leq r < \varepsilon.$$

Therefore, we have $\lim\limits_{\lambda \to 0} p_F(\lambda x) = 0$.

Assume now that $\lambda \in \mathbf{K}$ such that $|\lambda| \leq 1$, and moreover $x \in X$ and $\varepsilon > 0$. Then, by the definition of p_F, there exists an $r \in F^{-1}(x)$ such that $r < p_F(x) + \varepsilon$. Hence, since $x \in F(r)$ and $F(r)$ is balanced, it is clear that

$$\lambda x \in \lambda F(r) \subset F(r),$$

and thus $r \in F^{-1}(\lambda x)$. Therefore, we also have

$$p_F(\lambda x) = \inf \big(F^{-1}(\lambda x) \big) \leq r < p_F(x) + \varepsilon.$$

And hence, by letting $\varepsilon \to 0$, we can infer that $p_F(\lambda x) \leq p_F(x)$.

On the other hand, if $x,\, y \in X$, then again by the definition of p_F for each $\varepsilon > 0$ there exist $r \in F^{-1}(x)$ and $s \in F^{-1}(y)$ such that $r < p_F(x) + \varepsilon$ and $s < p_F(y) + \varepsilon$. Hence, by noticing that $x \in F(r)$ and $y \in F(s)$, and using the additivity property of F, we can infer that

$$x + y \in F(r) + F(s) \subset F(r+s),$$

and thus $r + s \in F^{-1}(x+y)$. Hence, it is clear that

$$p_F(x+y) = \inf \big(F^{-1}(x+y) \big) \leq r + s < p_F(x) + p_F(y) + 2\,\varepsilon,$$

and thus

$$p_F(x+y) \leq p_F(x) + p_F(y).$$

Therefore, p_F is a preseminorm on X.

Finally, if $r \in D_F$ and $x \in F_{p_F}(r)$, then by the corresponding definitions we have $x \in B_r^{p_F}(0)$, and hence $p_F(x) < r$. Therefore, by the definition of p_F, there exists an $s \in F^{-1}(x)$ such that $s < r$. Hence, by the monotonicity property of F, it is clear that $x \in F(s) \subset F(r)$. Therefore, $F_{p_F}(r) \subset F(r)$.

On the other hand, if $r \in D_F$ and $x \in F(r)$, then $r \in F^{-1}(x)$. Therefore, by the definition of p_F, we have $p_F(x) = \inf \big(F^{-1}(x) \big) \leq r$. Hence, by the corresponding definitions, it is clear that $x \in \bar{B}_r^{p_F}(0) = \overline{F}_{p_F}(r)$. Therefore, $F(r) \subset \overline{F}_{p_F}(r)$ is also true.

Now, as an immediate consequence of Theorem 3.2, we can also state

Corollary 3.3. *If F is a total preseminorm generating relation for X, then*

$$F_{p_F} \subset F \subset \overline{F}_{p_F}.$$

Moreover, in addition to Theorem 3.2, we can also easily prove

Theorem 3.4. *If p is a preseminorm on X and F is a preseminorm generating relation for X such that*

$$F_p(r) \subset F(r) \subset \overline{F}_p(r)$$

for all $r \in D_F$, then $p = p_F$.

Proof. If $x \in X$, then by the definition of p_F, for each $\varepsilon > 0$, there exists an $r \in F^{-1}(x)$ such that $r < p_F(x) + \varepsilon$. Hence, we can infer that

$$x \in F(r) \subset \overline{F}_p(r) = \bar{B}^p_r(0).$$

Therefore, $p(x) \le r < p_F(x) + \varepsilon$, and thus $p(x) \le p_F(x)$ is also true.

On the other hand, if $p(x) < p_F(x)$, then by the denseness property of D_F there exists an $r \in D_F$ such that $p(x) < r < p_F(x)$. Hence, we can infer that

$$x \in B^p_r(0) = F_p(r) \subset F(r).$$

Therefore, $r \in F^{-1}(x)$, and thus $p_F(x) = \inf\big(F^{-1}(x)\big) \le r < p_F(x)$. This contradiction shows that only $p(x) = p_F(x)$ can be true.

The $F = F_p$ and $F = \overline{F}_p$ particular cases of Theorem 3.4. immediately give

Corollary 3.5. *If p is a preseminorm on X, then*

$$p = p_{F_p} = p_{\,\overline{F}_p}.$$

4. THE KERNELS OF PRESEMINORM GENERATING RELATIONS

Definition 4.1. *If F is a preseminorm generating relation for X, then the set*

$$N_F = \bigcap_{r \in D_F} F(r)$$

will be called the kernel of F.

The appropriateness of the above definition is already apparent from the following

Theorem 4.2. *If p is a preseminorm on X, then*

$$N_{F_p} = N_{\,\overline{F}_p} = p^{-1}(0).$$

Proof. If $x \in p^{-1}(0)$, then $p(x) = 0$. Therefore, for each $r \in \mathbf{R}_+$, we have $p(x) < r$. Hence, by the corresponding definitions, it is clear that $x \in B^p_r(0) = F_p(r)$. Therefore, $x \in N_{F_p}$, and thus $p^{-1}(0) \subset N_{F_p}$.

On the other hand, if $x \in N_{\overline{F}_p}$, then by the corresponding definitions, for each $r \in \mathbf{R}_+$, we have $x \in \overline{F}_p(r) = \bar{B}^p_r(0)$, and hence $p(x) \leq r$. Therefore, $p(x) = 0$, and hence $x \in p^{-1}(0)$. Consequently, $N_{\overline{F}_p} \subset p^{-1}(0)$ is also true.

Now, since

$$N_{F_p} = \bigcap_{r \in \mathbf{R}_+} F_p(r) \subset \bigcap_{r \in \mathbf{R}_+} \overline{F}_p(r) = N_{\overline{F}_p},$$

it clear that the required equalities are also true.

Moreover, concerning the kernels of preseminorm generating relations, we can also easily prove the following

Theorem 4.3. *If F is a preseminorm generating relation for X, then*

$$N_F = p_F^{-1}(0).$$

Proof. By Theorems 4.2 and 3.2, it is clear that

$$p_F^{-1}(0) = N_{F_{p_F}} = \bigcap_{r \in \mathbf{R}_+} F_{p_F}(r) \subset \bigcap_{r \in D_F} F_{p_F}(r) \subset \bigcap_{r \in D_F} F(r) = N_F.$$

To prove converse inclusion, suppose now that $x \in N_F$ and $r \in \mathbf{R}_+$. Then, by the denseness property of D_F, there exists a $u \in D_F$ such that $u < r$. Hence, by using the assumption $x \in N_F$ and Theorems 3.2 and 2.9, we can infer that

$$x \in F(u) \subset \overline{F}_{p_F}(u) \subset \overline{F}_{p_F}(r).$$

Therefore, by Definition 4.1 and Theorem 4.2, we also have $x \in N_{\overline{F}_{p_F}} = p_F^{-1}(0)$.

From Theorem 4.3, since the kernel $p^{-1}(0)$ of a preseminorm p on X is a linear subspace of X, it is clear that in particular we also have

Corollary 4.4. *If F is a preseminorm generating relation for X, then N_F is a linear subspace of X.*

Definition 4.5. *A preseminorm generating relation F for X will be called separating if $N_F = \{0\}$.*

Remark 4.6. Note that, because of $0 \in N_F$, the preseminorm generating relation F is separating if and only if $N_F \subset \{0\}$.

That is, for each $x \in X$, with $x \neq 0$, there exists an $r \in D_F$ such that $x \notin F(r)$, while $0 \in F(r)$ automatically holds.

Moreover, as some immediate consequences of Theorems 4.3 and 4.2, we can also state the following two theorems.

Theorem 4.7. *If F is a preseminorm generating relation for X, then the following assertions are equivalent:*

(1) *F is separating*; (2) *p_F is a prenorm.*

Theorem 4.8. *If p is a preseminorm on X, then the following assertions are equivalent:*

(1) F_p *is separating*; (2) $\overline{F}_p$ *separating*; (3) p *is a prenorm.*

By the corresponding definitions, we evidently have the following

Theorem 4.9. *If D is an admissible subset of $\mathbf{R}_+$ and G is a preseminorm generating relation for X such that $D \subset D_G$, then $F = G \mid D$ is also a preseminorm generating relation for X.*

HINT. To check that $X = F[D]$, note that if $x \in X$, then since $X = G[D_G]$ there exists an $r \in D_G$ such that $x \in G(r)$. Moreover, since D is dense in $\mathbf{R}_+$, there exists a $v \in D$ such that $r < v$. Hence, by the monotonicity property of G it is clear that $x \in G(r) \subset G(v) = F(v)$.

Moreover, concerning the restrictions of preseminorm generating relations, we can also easily prove the following

Theorem 4.10. *If F and G are preseminorm generating relations for X such that $F = G \mid D_F$, then $p_F = p_G$.*

Proof. In this case, by Theorem 3.2, we have $F_{p_G}(r) \subset F(r) \subset \overline{F}_{p_G}(r)$ for all $r \in D_F$. Therefore, by Theorem 3.4, the required equality is also true.

From Theorem 4.10, by Theorem 4.3, it is clear that in particular we also have

Corollary 4.11. *If F and G are preseminorm generating relations for X such that $F = G \mid D_F$, then $N_F = N_G$.*

REMARK 4.12. Thus, in particular, F is separating if and only if G is separating.

5. OPERATIONS ON PRESEMINORM GENERATING RELATIONS

Definition 5.1. *If F is a preseminorm generating relation for X, and F_* and F^* are relations on $\mathbf{R}_+$ to X such that*

$$F_*(r) = \bigcup_{u<r} F(u) \qquad \text{and} \qquad F^*(r) = \bigcap_{r<v\in D_F} F(v)$$

for all $r \in \mathbf{R}_+$, then the relations F_* and F^* will be called the lower and upper regularizations of F, respectively.

The importance of the above definition is apparent from the following

Theorem 5.2. *If F is a preseminorm generating relation for X, then*

$$F_* = F_{p_F} \qquad \text{and} \qquad F^* = \overline{F}_{p_F}.$$

Proof. If $r \in \mathbf{R}_+$ and $x \in F_*(r)$, then by the definition of F_* there exists a $u \in D_F$ such that $u < r$ and $x \in F(u)$. Hence, it is clear that $u \in F^{-1}(x)$, and thus $p_F(x) = \inf\big(F^{-1}(x)\big) \leq u < r$. Therefore, $x \in B_r^{p_F}(0) = F_{p_F}(r)$, and thus $F_*(r) \subset F_{p_F}(r)$.

Moreover, if $x \in F_{p_F}(r)$, then by the corresponding definitions $x \in B_r^{p_F}(0)$, and hence $p_F(x) < r$. Therefore, by the definition of p_F, there exists a $u \in F^{-1}(x)$ such that $u < r$. Hence, by noticing that $x \in F(u)$, we can see that $x \in \bigcup_{u<r} F(u) = F_*(r)$. Therefore, $F_{p_F}(r) \subset F_*(r)$ is also true.

On the other hand, if $r \in \mathbf{R}_+$ and $x \in X$ such that $x \notin \overline{F}_{p_F}(r)$, then by the corresponding definitions $x \notin \bar{B}_r^{p_F}(0)$, and hence $r < p_F(x)$. Moreover, by the denseness property of D_F, there exists a $v \in D_F$ such that $r < v < p_F(x)$. Hence, by the definition of p_F, it is clear that $v \notin F^{-1}(x)$, and thus $x \notin F(v)$. Therefore, $x \notin \bigcap_{r<v\in D_F} F(v) = F^*(r)$, and thus $F^*(r) \subset \overline{F}_{p_F}(r)$.

Moreover, if $x \in \overline{F}_{p_F}(r)$, then by the corresponding definitions $x \in \bar{B}_r^{p_F}(0)$, and hence $p_F(x) \leq r$. Therefore, by the definition of p_F, for each $v \in D_F$, with $r < v$ there exists a $u \in F^{-1}(x)$ such that $u < v$. Hence, by the monotonicity property of F, it is clear that $x \in F(u) \subset F(v)$. Therefore, $x \in \bigcap_{r<v\in D_F} F(v) = F^*(r)$, and thus $\overline{F}_{p_F}(r) \subset F^*(r)$ is also true.

From Theorem 5.2, by Theorem 3.2, it is clear that in particular we also have

Corollary 5.3. *If F is a preseminorm generating relation for X, then F_* and F^* are total preseminorm generating relations for X such that $F_* \subset F^*$ and*

$$F_*(r) \subset F(r) \subset F^*(r)$$

for all $r \in D_F$.

In addition to Definition 5.1, we may also naturally introduce the following

Definition 5.4. *If F is a preseminorm generating relation for X, and $\widetilde{F}$ is a relation on $\mathbf{R}_+$ to X such that*

$$\widetilde{F}(r) = \bigcup_{u\leq r} F(u)$$

for all $r \in \mathbf{R}_+$, then the relation $\widetilde{F}$ will be called the natural extension of F.

The appropriateness of the above definition is apparent from the following

Theorem 5.5. *If F is a preseminorm generating relation for X, then $\widetilde{F}$ is a preseminorm generating relation for X such that*

$$\widetilde{F}(r) = F(r) \quad \text{for} \quad r \in D_F \quad \text{and} \quad \widetilde{F}(r) = F_*(r) \quad \text{for} \quad r \in \mathbf{R}_+ \setminus D_F.$$

Hint. To prove the required additivity property of $\widetilde{F}$, let $r, s \in \mathbf{R}_+$, and assume $x \in \widetilde{F}(r)$ and $y \in \widetilde{F}(s)$. Then, by the definition of $\widetilde{F}$, there exist $u, v \in D_F$, with $u \leq r$ and $v \leq s$, such that $x \in F(u)$ and $y \in F(v)$. Hence, by the additivity property of F, it follows that $x + y \in F(u) + F(v) \subset F(u + v)$. Now, since $u + v \in D_F$ such that $u + v \leq r + s$, it is clear that $x + y \in \widetilde{F}(r + s)$. Therefore, $\widetilde{F}(r) + \widetilde{F}(s) \subset \widetilde{F}(r + s)$.

From Theorem 5.5, by Corollary 5.3, it is clear that in particular we also have

Corollary 5.6. *If F is a preseminorm generating relation for X, then*

$$F_* \subset \widetilde{F} \subset F^*.$$

Concerning the operations given in Definitions 5.1 and 5.4, we can also easily prove the following two theorems.

Theorem 5.7. *If F is a preseminorm generating relation for X, then*

$$p_F = p_{F_*} = p_{\widetilde{F}} = p_{F^*}.$$

Proof. By Theorem 5.2 and Corollary 5.6, we have $F_{p_F} = F_* \subset \widetilde{F} \subset F^* = \overline{F}_{pF}$. And hence, by Theorem 3.4, it is clear that the required equalities are also true.

From Theorem 5.7, by Theorem 4.3, it is clear that in particular we also have

Corollary 5.8. *If F is a preseminorm generating relation for X, then*

$$N_F = N_{F_*} = N_{\widetilde{F}} = N_{F^*}.$$

Theorem 5.9. *If F and G are preseminorm generating relations for X such that $F = G \,|\, D_F$, then $F_* = G_*$ and $F^* = G^*$.*

Proof. In this case, by Theorem 4.10, we have $p_F = p_G$. Hence, by Theorem 5.2, it is clear that the required equalities are also true.

From Theorem 5.9, by Theorem 5.5, it is clear that in particular we also have

Corollary 5.10. *If F is a preseminorm generating relation for X, then*

$$F_* = \big(\widetilde{F}\big)_* \quad \text{and} \quad F^* = \big(\widetilde{F}\big)^*.$$

6. LOWER AND UPPER REGULAR PRESEMINORM GENERATING RELATIONS

Definition 6.1. *A preseminorm generating relation F for X will be called lower (upper) regular if $F(r) = F_*(r)$ $\big(F(r) = F^*(r)\big)$ for all $r \in D_F$.*

Remark 6.2. Note that, by Corollary 5.3, the preseminorm generating relation F is lower (upper) regular if and only if $F(r) \subset F_*(r)$ $\big(F^*(r) \subset F(r)\big)$ for all $r \in D_F$.

Moreover, as some immediate consequences of Theorem 5.2, we can at once state the following two theorems.

Theorem 6.3. *If F is a preseminorm generating relation for X, then the following assertions are equivalent:*

(1) *F is lower regular;* (2) *$F(r) = F_{p_F}(r)$, for all $r \in D_F$.*

Theorem 6.4. *If F is a preseminorm generating relation for X, then the following assertions are equivalent:*

(1) F *is upper regular*; (2) $F(r) = \overline{F}_{p_F}(r)$, *for all* $r \in D_F$.

Furthermore, from Theorem 5.2, by using Corollary 3.5, we can also at once get the following two theorems.

Theorem 6.5. *If p is a preseminorm on X, then*

(1) F_p *is lower regular*; (2) $\overline{F}_p$ *is upper regular.*

Proof. By Corollary 3.5 and Theorem 5.2, we have

$$F_p = F_{p_{F_p}} = \big(F_p\big)_* \quad \text{and} \quad \overline{F}_p = \overline{F}_{p_{\overline{F}_p}} = \big(\overline{F}_p\big)^*.$$

Hence, by Theorem 5.2, it is clear that in particular we also have

Corollary 6.6. *If F is a preseminorm generating relation for X, then*

(1) F_* *is lower regular*; (2) F^* *is upper regular.*

Theorem 6.7. *If p is a preseminorm on X, then*

$$F_p = \big(\overline{F}_p\big)_* \quad \textit{and} \quad \overline{F}_p = \big(F_p\big)^*.$$

Proof. By Corollary 3.5 and Theorem 5.2, we also have

$$F_p = F_{p_{\overline{F}_p}} = \big(\overline{F}_p\big)_* \quad \text{and} \quad \overline{F}_p = \overline{F}_{p_{F_p}} = \big(F_p\big)^*.$$

Hence, by Theorem 5.2, it is clear that in particular we also have

Corollary 6.8. *If F is a preseminorm generating relation for X, then*

$$F_* = \big(F^*\big)_* \quad \textit{and} \quad F^* = \big(F_*\big)^*.$$

Now, in addition to Theorem 6.5, we can also easily prove the following

Theorem 6.9. *If p is a preseminorm on X, then the following assertions are equivalent:*

(1) F_p *is upper regular*; (2) $\overline{F}_p$ *is lower regular*;

(3) $F_p = \overline{F}_p$; (4) $p = 0$; (5) $F_p = \mathbf{R}_+ \times X$; (6) $\overline{F}_p = \mathbf{R}_+ \times X$.

Proof. If the assertions (1) and (2) hold, then by Definition 6.1 and Theorem 6.7 we have $F_p = \big(F_p\big)^* = \overline{F}_p$ and $\overline{F}_p = \big(\overline{F}_p\big)_* = F_p$, respectively. Therefore, the implications (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) are true.

Moreover, if the assertion (4) does not hold, then there exists an $x \in X$ such that $p(x) \neq 0$. Now, by defining $r = p(x)$, we can see that $r \in \mathbf{R}_+$ such that

$x \in \bar{B}_r^p(0) = \overline{F}_p(r)$, but $x \notin B_r^p(0) = F_p(0)$. Therefore, the assertion (3) does not also hold. Thus, the implication (3) $\Rightarrow$ (4) is also true.

On the other hand, if the assertion (4) holds, then we evidently have $F_p(r) = B_r^0(r) = X$, and thus the assertion (5) also holds. Moreover, if the assertion (5) holds, then since $F_p \subset \overline{F}_p$ it is clear that the assertion (6), and thus the assertion (3) also holds.

Finally, to complete the proof, we note that if the assertion (3) holds, then by Theorem 6.5 it is clear that the assertion (1) also holds. Moreover, if the assertion (6) holds, then by the corresponding definitions it is clear that the assertion (2) also holds.

From Theorem 6.9, by Theorem 5.2, it is clear that in particular we also have

Corollary 6.10. *If F is a preseminorm generating relation for X, then the following assertions are equivalent*:

(1) F_* *is upper regular*; (2) F^* *is lower regular*;

(3) $F_* = F^*$; (4) $p_F = 0$; (5) $F_* = \mathbf{R}_+ \times X$; (6) $F^* = \mathbf{R}_+ \times X$;

(7) $F = D_F \times X$.

HINT. To prove the equivalence of the assertions (6) and (7), note that if $r \in D_F$, then by the denseness property of $\mathbf{R}_+$ there exists an $s \in \mathbf{R}_+$ such that $s < r$. Therefore, if the assertion (6) holds, then we have $X = F^*(s) \subset F(r)$, and hence $F(r) = X$.

Finally, we note that as an immediate consequence of Theorem 5.9, we can also state

Theorem 6.11. *If F and G are preseminorm generating relations for X such that $F = G \,|\, D_F$ and G is lower (upper) regular, then F is also lower (upper) regular.*

Hence, by Theorem 5.5, it is clear that in particular we also have

Corollary 6.12. *If F is a preseminorm generating relation for X such that $\widetilde{F}$ is lower (upper) regular, then F is also lower (upper) regular.*

However, the latter assertion is of no particular importance since we also have the following

Theorem 6.13. *A preseminorm generating relation F for X, then the following assertions are equivalent*:

(1) F *is lower regular*; (2) $F_* = \widetilde{F}$; (3) $\widetilde{F}$ *is lower regular.*

Proof. To check this, note that the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) follow immediately from Theorem 5.5 and Corollaries 6.6 and 6.12, respectively.

7. SEMINORM GENERATING RELATIONS AND THEIR BASIC PROPERTIES

Definition 7.1. *A relation F of a regular admissible subset D_F, of $\mathbf{R}_+$ onto X will be called a seminorm generating relation for X if*

(1) $F(r) + F(s) \subset F(r+s)$ *for all* $r,\, s \in D_F$;

(2) $\lambda F(r) \subset F(tr)$ *for all* $\lambda \in \mathbf{K}$ *and* $r,\, t \in D_F$ *with* $|\lambda| \leq t$.

For a preliminary illustration of seminorm generating relations, we can at once state

Example 7.2. If p is a seminorm on X, then F_p and $\overline{F}_p$ are seminorm generating relations for X.

Moreover, to reveal the relationship between seminorm and preseminorm generating relations, we can easily prove

Theorem 7.3. *If F is a relation of a regular admissible subset D_F of $\mathbf{R}_+$ to X, then the following assertions are equivalent:*

(1) *F is a seminorm generating relation for X;*

(2) *F is a homogeneous preseminorm generating relation for X.*

Proof. If the assertion (2) holds, then by the assumed balancedness and homogenity property of F it is clear that

$$\lambda F(r) \subset tF(r) \subset F(tr)$$

for all $\lambda \in \mathbf{K}$ and $r,\, t \in D_F$ with $|\,\lambda\,| \leq t$. Therefore, the assertion (1) also holds.

To prove the converse implication, suppose now that the assertion (1) holds, and let $r \in D_F$ and $x \in X$. Then, since $X = F[D_F]$, there exists an $s \in D_F$ such that $x \in F(s)$. Moreover, since D_F is dense in $\mathbf{R}_+$, there exists a $t \in D_F$ such that $t < rs^{-1}$, and thus $ts < r$. Hence, by the assumed homogenity and additivity properties of F, it is clear that

$$tx \in tF(s) \subset F(ts) = F(ts) + 0\, F(r-ts) \subset F(ts) + F(r-ts) \subset F(r).$$

Therefore, $x \in t^{-1}F(r)$, and thus $F(r)$ is an absorbing subset of X. Now, since the balancedness property of $F(r)$ is immediate from the assumed homogenity property of F, is clear that the assertion (2) also holds.

From Theorem 7.3, by Theorem 4.9, it is clear that in particular we also have

Corollary 7.4. *If D is a regular admissible subset of $\mathbf{R}_+$ and G is a seminorm generating relation for X, then $F = G\,|\,D$ is also a seminorm generating relation for X.*

Moreover, as an immediate consequence of the corresponding definitions, we can also state

Theorem 7.5. *If F is a seminorm generating relation for X such that $1 \in D_F$, then F is D_F–convex.*

Proof. By the corresponding properties of D_F and F, we have

$$tF(r) + (1-t)\,F(s) \subset F(tr) + F((1-t)s) \subset F(tr + (1-t)s)$$

for all r, s, $t \in D_F$ with $t < 1$.

The $r = s$ particular case of the above inclusion immediately gives

Corollary 7.6. *If F is a seminorm generating relation for X such that $1 \in D_F$, then $F(r)$ is D_F–convex for all $r \in D_F$.*

Now, as useful characterization of seminorm generating relations, we can also easily prove

Theorem 7.7. *If F is a relation of a normal admissible subset D_F of $\mathbf{R}_+$ to X, then the following assertions are equivalent*:

(1) *F is a seminorm generating relation for X*;

(2) *there exists an absorbing, balanced, D_F–convex subset A of X such that $F(r) = rA$ for all $r \in D_F$.*

Proof. If the assertion (1) holds and r, $s \in D_F$, then by the homogenity property of F we have

$$rF(s) \subset F(rs).$$

Hence, by writing r^{-1} in place of r, and rs in place of s, we can see that

$$r^{-1}F(rs) \subset F(s).$$

This implies that $F(rs) \subset rF(s)$. Therefore, the equality

$$F(rs) = rF(s)$$

is also true. Hence, under the notation $A = F(1)$, it follows that

$$F(r) = rF(1) = rA.$$

Moreover, from Theorem 7.3 and Corollary 7.6, we can see that A is an absorbing, balanced, D_F–convex subset of X. Therefore, the assertion (2) also holds.

The proof of converse implication (2) $\Rightarrow$ (1) is much more obvious. Moreover, it shows, in particular, that the following assertion is also true.

Corollary 7.8. *If F is a seminorm generating relation for X such that D_F is normal admissible subset of $\mathbf{R}_+$, then*

$$F(rs) = rF(s) \quad \text{and} \quad F(r+s) = F(r) + F(s)$$

for all r, $s \in D_F$.

The appropriateness of Definition 7.1, is also apparent from the following

Theorem 7.9. *If F is a seminorm generating relation for X, then p_F is a seminorm on X.*

Proof. Because of Theorems 7.3 and 3.2, we need only prove the required homogenity property of p_F. For this, assume that $\lambda \in \mathbf{K}$ and $x \in X$, and moreover $\varepsilon > 0$. Then, by the denseness property of D_F, there exists a $t \in D_F$ such that $|\lambda| < t < |\lambda| + \varepsilon$. Moreover, by the definition of p_F, there exists an $r \in F^{-1}(x)$ such that $r < p_F(x) + \varepsilon$. Hence, by noticing that $x \in F(r)$ and using the homogenity property of F, we can infer that

$$\lambda x \in \lambda F(r) \subset F(tr),$$

and thus $tr \in F^{-1}(\lambda x)$. Now, it is clear that

$$p_F(\lambda x) = \inf\big(F^{-1}(\lambda x)\big) \leq tr < \big(|\lambda| + \varepsilon\big)\big(p_F(x) + \varepsilon\big).$$

And hence, by letting $\varepsilon \to 0$, we can infer that $p(\lambda x) \leq |\lambda| p_F(x)$.

From Theorem 7.9, by Theorem 5.2, it is clear that in particular we also have

Corollary 7.10. *If F is a seminorm generating relation for X, then F_* and F^* are also seminorm generating relations for X.*

8. A FEW ILLUSTRATING EXAMPLES

The following example shows, in particular, that the converse of the second assertion of Corollary 6.12 and a hoped for counterpart of Theorem 5.5 for seminorm generating relations are not true.

Example 8.1. If F is a relation on $\mathbf{Q}_+$ to $\mathbf{R}$ such that

$$F(r) = [-r,\, r]$$

for all $r \in \mathbf{Q}_+$, then F is an upper, but not lower regular, separating seminorm generating relation for $\mathbf{R}$ such that $\widetilde{F}$ is neither homogeneous nor upper or lower regular.

Note that the function p, defined by $p(x) = |x|$ for all $x \in \mathbf{R}$, is a norm on $\mathbf{R}$ such that $F = \overline{F}_p \,|\, \mathbf{Q}_+$. Therefore, by Example 7.2 and Corollary 7.4, F is a seminorm generating relation for X. Moreover, by Theorem 4.8 and Remark 4.12, F is separating.

On the other hand, by Theorems 5.9, 6.5 and 6.7, we have

$$F^* = \big(\overline{F}_p \,|\, \mathbf{Q}_+\big)^* = \big(\overline{F}_p\big)^* = \overline{F}_p \quad \text{and} \quad F_* = \big(\overline{F}_p \,|\, \mathbf{Q}_+\big)_* = \big(\overline{F}_p\big)_* = F_p.$$

Therefore,

$$F^*(r) = \overline{F}_p(r) = [-r,\, r] = F(r),$$

and

$$F_*(r) = F_p(r) =]-r,\, r\,[\neq [-r,\, r] = F(r)$$

for all $r \in \mathbf{Q}_+$. And thus, F is upper, but not lower regular.

Moreover, from the definition of F and the equality $F_* = F_p$, by Theorem 5.5, it is clear that

$$\widetilde{F}(r) = [-r,\, r] \quad \text{if} \quad r \in \mathbf{Q}_+ \quad \text{and} \quad \widetilde{F}(r) =]-r,\, r\,[\quad \text{if} \quad r \in \mathbf{R}_+ \setminus \mathbf{Q}_+ .$$

Therefore,

$$\widetilde{F}(r) =]-r,\, r\,[\neq [-r,\, r] = r[-1,\, 1] = r\,\widetilde{F}(1)$$

for all $r \in \mathbf{R}_+ \setminus \mathbf{Q}_+$. And thus, by Theorems 7.7 and 7.3, $\widetilde{F}$ is not homogeneous.

Finally, from Corollary 5.10 and the above equalities, it is clear that

$$\big(\widetilde{F}\big)_*(r) = F_*(r) = F_p(r) =]-r,\, r\,[\neq [-r,\, r] = \widetilde{F}(r)$$

for all $r \in \mathbf{Q}_+$, and

$$\big(\widetilde{F}\big)^*(r) = F^*(r) = \overline{F}_p(r) = [-r,\, r] \neq]-r,\, r\,[= \widetilde{F}(r)$$

for all $r \in \mathbf{R}_+ \setminus \mathbf{Q}_+$. And thus, $\widetilde{F}$ is neither lower nor upper regular.

Remark 8.2. If F and G are preseminorm generating relations for X such that $F = G \,|\, D_F$, then in contrast to Theorem 5.9 we can only state that $\widetilde{F} \subset \widetilde{G}$ such that $\widetilde{F}(r) = \widetilde{G}(r)$ for $r \in D_F \cup (\mathbf{R}_+ \setminus D_)$. Namely, by the following example, the equality of $\widetilde{F}$ and $\widetilde{G}$ on $D_G \setminus D_F$ need not be true.

Example 8.3. If F is as in Example 8.1 and G is a relation on $\mathbf{R}_+$ to $\mathbf{R}$ such that

$$G(r) = [-r,\, r]$$

for all $r \in \mathbf{R}_+$, then F and G are upper, but not lower regular, separating seminorm generating relations for X, with $F = G \,|\, \mathbf{Q}_+$, such that $\widetilde{F}(r) \neq \widetilde{G}(r)$ for all $r \in \mathbf{R}_+ \setminus \mathbf{Q}_+$.

To see the latter assertion, note that by Example 8.1

$$\widetilde{F}(r) = [-r,\, r] \quad \text{if} \quad r \in \mathbf{Q}_+ \quad \text{and} \quad \widetilde{F}(r) =]-r,\, r\,[\quad \text{if} \quad r \in \mathbf{R}_+ \setminus \mathbf{Q}_+ .$$

While, by Theorem 5.5, $\widetilde{G}(r) = [-r,\, r]$ for all $r \in \mathbf{R}_+$.

Remark 8.4. If F is a preseminorm generating relation for X, and $\widetilde{F}$ is a relation on $\mathbf{R}_+$ to X such that

$$\widetilde{F}(r) = \bigcap_{r \leq v \in D_F} F(v)$$

for all $r \in \mathbf{R}_+$, then in contrast to Corollary 5.3 and Theorem 5.5 we can only state that $\widetilde{F}$ is an absorbing, balanced valued relation of $\mathbf{R}_+$ onto X such that

$$\widetilde{F}(r) = F(r) \quad \text{for} \quad r \in D_F \quad \text{and} \qquad \widetilde{F}(r) = F^*(r) \quad \text{for} \quad r \in \mathbf{R}_+ \setminus D_F ,$$

and thus $F_* \subset \widetilde{F} \subset \widetilde{F} \subset F^*$. Namely, by the following example, the relation $\widetilde{F}$ need not have the additivity property of preseminorm generating relations.

EXAMPLE 8.5. If F is a relation on $\mathbf{Q}_+$ to $\mathbf{R}$ such that

$$F(r) =]-r, r[$$

for all $r \in \mathbf{Q}_+$, then F is a lower, but not upper regular, separating seminorm generating relation for $\mathbf{R}$ such that

$$\widetilde{F}(r) + \widetilde{F}(s) \not\subset \widetilde{F}(r+s)$$

for all $r, s \in \mathbf{R}_+ \setminus \mathbf{Q}_+$ with $r + s \in \mathbf{Q}_+$.

To prove the latter assertion, note that now, for the norm p defined in Example 8.1, we have $F = F_p \,|\, \mathbf{Q}_+$. Moreover, by Theorems 5.9 and 6.7, we now have

$$F^* = \big(F_p \,|\, \mathbf{Q}_+\big)^* = \big(F_p\big)^* = \overline{F}_p.$$

And thus, by Remark 8.4, we also have

$$\widetilde{F}(r) =]-r, r[\quad \text{if} \quad r \in \mathbf{Q}_+ \quad \text{and} \quad \widetilde{F}(r) = [-r, r] \quad \text{if} \quad r \in \mathbf{R}_+ \setminus \mathbf{Q}_+.$$

Therefore, if $r, s \in \mathbf{R}_+ \setminus \mathbf{Q}_+$ such that $r + s \in \mathbf{Q}_+$, then

$$\widetilde{F}(r) + \widetilde{F}(s) = [-r, r] + [-s, s] = [-(r-s), r+s],$$

while $\widetilde{F}(r+s) =]-(r-s), r+s[$.

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