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ON APPROXIMATIVE SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS BY MEANS OF BERNSTEIN POLYNOMIALS

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In this paper, we give the estimation of the approximation of the differential equation solution

$$y'' = f(x,y), \quad y(0) = y'(0) = 0, \quad x \in [0,h],$$

where $f(x,y) \in AN^1$, by the use of the Bernstein's polynomial

$$B_n\big[\varphi(t);x\big] = \frac{1}{h^n}\sum_{k=0}^{n} \varphi\big(k\,\frac{h}{n}\big)\binom{n}{k} x^k (h-x)^{n-k},$$

where $\varphi(t) \in L^\infty[0,h]$.

Let $D = \{(x,y) \,:\, 0 \le x \le h,\ |y| \le a\}$. Denote by AN^1 (see [**1**]) the class of all functions $f(x,y)$ such that $f(x,y)$ is countinuous on D and by the second coordinate satisfying the Lipschitz condition with the constant A:

$$|f(x,y_1) - f(x,y_2)| \le A\ |y_1 - y_2|.$$

On the segment $[0,h]$, Bernstein polynomials are of the form:

$$B_n\big[\varphi(t);x\big] = \frac{1}{h^n}\sum_{k=0}^{n} \varphi\big(k\,\frac{h}{n}\big)\binom{n}{k} x^k (h-x)^{n-k},$$

where $\varphi(t) \in L^\infty[0,h]$.

We consider the equation of the following form

$$y'' = f(x,y), \quad y(0) = y'(0) = 0, \quad x \in [0,h], \tag{1}$$

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where $f(x,y) \in AN^1$. We give this equation (1) in the integral form:

$$y(x) = \int_0^x (x-t)\, f(t, y(t))\, \mathrm{d}t.$$

The method for the approximate solution of ordinary differential equations by the use of linear operators was introduced by V. K. Dziadik in [**1**]. By following that method, we take Bernstein polynomials, as an example of linear operators, and we look for the approximate solution of the equation (1) by the use of the equation

$$\tilde{y}_n(x) = B_n\left[\int_0^{\xi} (\xi - t)\, f[t, \tilde{y}_n(t)]\, \mathrm{d}t; x\right]. \tag{2}$$

We give the result, without the proof, given by B. B. Krochuk in [**2**], about the approximation of the solution of the differential equation (1), where Bernstein polynomials are used.

Theorem A. *For the arbitrary equation of the form* (1), *where* $f(x,y) \in AN^1$ *and arbitrary number* h, *such that the equation* (2) *is solvable on the segment* $[0,h]$, *the polynomial* $\tilde{y}_n(x)$ *which is the solution of that equation, approximates on* $[0,h]$ *the solution* $y(x)$ *of the equation* (1) *and the inequality*

$$\|y(x) - \tilde{y}_n(x)\|_C \le (1+\alpha_n)\big\|y(x) - B_n\,[y(t);x]\,\big\|_C\, e^{Ah^2} \tag{3}$$

is valid, where

$$\alpha_n = \frac{Ah^2 e^{Ah^2}}{8n - Ah^2 e^{Ah^2}}.$$

In this paper, we get different results. In the proof we use the following characteristic of the Bernstein polynomial.

Lema 1. *Let* $\varphi_1(x),\ \varphi_2(x) \in C_{[0,h]}$. *Then, for each* $n \in \mathbf{N}$, *holds*:

$$\Big|B_n[\varphi_1(t);x] - B_n[\varphi_2(t);x]\Big| \le \|\varphi_1(x) - \varphi_2(x)\|_C.$$

Proof.

$$\begin{aligned}\Big|B_n[\varphi_1(t);x] - B_n[\varphi_2(t);x]\Big| &= \Bigg|\frac{1}{h^n}\sum_{k=0}^{n}\varphi_1\Big(k\,\frac{h}{n}\Big)\binom{n}{k}x^k(h-x)^{n-k} \\ &\qquad - \frac{1}{h^n}\sum_{k=0}^{n}\varphi_2\Big(k\,\frac{h}{n}\Big)\binom{n}{k}x^k(h-x)^{n-k}\Bigg| \\ &= \Bigg|\frac{1}{h^n}\sum_{k=0}^{n}\Big[\varphi_1\Big(k\frac{h}{n}\Big) - \varphi_2\Big(k\frac{h}{n}\Big)\Big]\binom{n}{k}x^k(h-x)^{n-k}\Bigg|\end{aligned}$$

$$\le \frac{1}{h^n}\sum_{k=0}^{n}\Big|\varphi_1\Big(k\,\frac{h}{n}\Big)-\varphi_2\Big(k\,\frac{h}{n}\Big)\Big|\binom{n}{k}x^k(h-x)^{n-k}$$
$$\le \|\varphi_1(x)-\varphi_2(x)\|\,\frac{1}{h^n}\sum_{k=0}^{n}\binom{n}{k}x^k(h-x)^{n-k}$$
$$= \|\varphi_1(x)-\varphi_2(x)\|. \qquad \square$$

Now, we give the main result of this paper.

Theorem 1. *For the arbitrary equation of the form* (1)*, where* $f(x,y)\in AN^1$ *and arbitrary number* h*, such that the equation* (2) *is solvable on the segment* $[0,h]$*, the polynomial* $\tilde{y}_n(x)$*, which is the solution of that equation, approximates on* $[0,h]$ *the solution* $y(x)$ *of the equation* (1) *and the inequality*

$$\|y(x)-\tilde{y}_n(x)\|_C \le (1+\alpha)\|y(x)-B_n[y(t);x]\|_C \tag{4}$$

is valid, where

$$\alpha=\begin{cases}\dfrac{\eta}{2-\eta}, & \eta\overset{\text{def}}{=}Ah^2<2\\ \infty, & \eta\ge 2.\end{cases}$$

Proof.

$$\begin{aligned}
|y(x)-\tilde{y}_n(x)| &= \big|y(x)-B_n[y(t);x]+B_n[y(t);x]-\tilde{y}_n(x)\big|\\
&\le \big|y(x)-B_n[y(t);x]\big|\\
&+\left|B_n\Big[\int_0^{\xi}(\xi-t)f[t,y(t)]\,\mathrm{d}t;x\Big]-B_n\Big[\int_0^{\xi}(\xi-t)f[t,\tilde{y}_n(t)]\,\mathrm{d}t;x\Big]\right|\\
&\le \big|y(x)-B_n[y(t);x]\big|\\
&+\left\|\int_0^{x}(x-t)f[t,y(t)]dt-\int_0^{x}(x-t)f[t,\tilde{y}_n(t)]\mathrm{d}t\right\|\\
&= \big|y(x)-B_n[y(t);x]\big|+\left\|\int_0^{x}(x-t)\big(f[t,y(t)]-f[t,\tilde{y}_n(t)]\big)\,\mathrm{d}t\right\|\\
&\le \big\|y(x)-B_n[y(t);x]\,\big\|+\int_0^{h}(h-t)\big\|f[t,y(t)]-f[t,\tilde{y}_n(t)]\big\|\,\mathrm{d}t\\
&\le \big\|y(x)-B_n[y(t);x]\,\big\|+A\big\|y(t)-\tilde{y}_n(t)\big\|\int_0^{h}(h-t)\,\mathrm{d}t\\
&= \big\|y(x)-B_n[y(t);x]\,\big\|+A\,\big\|y(t)-\tilde{y}_n(t)\big\|\,\frac{h^2}{2}.
\end{aligned}$$

Since the previous calculation holds for every $x\in[0,h]$, it follows that:

$$\big\|y(x)-\tilde{y}_n(x)\big\|\le\big\|y(x)-B_n[y(t);x]\,\big\|+\frac{Ah^2}{2}\big\|y(x)-\tilde{y}_n(x)\big\|,$$

and therefore

$$\big\|y(x)-\tilde{y}_n(x)\big\| \le \frac{1}{1-\dfrac{Ah^2}{2}}\,\big\|y(x)-B_n[y(t);x]\big\|.$$

This proves the theorem. □

Remark. Operational equation (2) is solvable for $Ah^2 < 2$ (see [**2**]).

Next we compare the results (3) and (4). Notice that

$$e^{Ah^2}\,\big\|y(x)-B_n[y(t);x]\big\| \le (1+\alpha_n)\big\|y(x)-B_n[y(t);x]\big\|\,e^{Ah^2},$$

for $\alpha_n \ge 0$ $(n \ge 2)$. Compare the left side with the right side of the inequality (4). What is larger: e^{Ah^2} or $\dfrac{1}{1-\dfrac{Ah^2}{2}}$? Let $x = Ah^2$, $g(x) = e^x - \dfrac{1}{1-\dfrac{x}{2}}$ and $0 < x < 2$. Then it is easy to prove that $g(x) > 0$ for $x \in]0,\xi_0[$ where $\xi_0 \approx 1,60$.

Therefore, we get that e^{Ah^2} is greater than $\dfrac{1}{1-\dfrac{Ah^2}{2}}$ for $Ah^2 \in]0,\xi_0[$ which shows that the estimation (4) is better than estimation (3) on the interval $]0,\xi_0[$.

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