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TRANSFER MATRICES OF N-DIMENSIONAL SIERPINSKI TETRAHEDRON

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We present the transfer matrix $M = (m_i^j)$ of n-dimensional Sierpinski tetrahedron and show that the Fractal dimension of the n-dimensional Sierpinski tetrahedron is equal to $\ln(n+1)/\ln 2$ in connection with the matrix $M = (m_i^j)$.

1. INTRODUCTION

Mandelbrot, Gefent, Aharony and Peyriere [**4**] introduced *transfer matrices of fractals*. In [**4**], it was shown that when two transfer matrices of a fractal coming from related geometric constructions are diagonalizable.

Wen [**6**] considered also diagonalizability of transfer matrices of fractals. Nenska–Ficek [**3**] considered *duality of fractals* and the dual of Sierpinski gasket. Kim–Kim [**2**] studied the fractal dimension of an n-dimensional Sierpinski tetrahedron.

In this paper we construct transfer matrices of n-dimensional Sierpinski tetrahedron, denoting it by $\triangle_n$, and discuss the fractal dimension of $\triangle_n$. We also consider the dual Sierpinski gasket.

2. AN EXAMPLE OF TRANSFER MATRIX OF SIERPINSKI GASKET

In this section, we give an example of transfer matrix of Sierpinski gasket.

(1) We define two sets $S = \{1, 2, 3\}$ and $E = \{w, e, s\}$. We define two mappings τ and ϕ as follows: $\tau(1) = \{s\}$, $\tau(2) = \{e\}$, $\tau(3) = \{w\}$, $\phi(1) = \{w, e\}$, $\phi(2) = \{w, s\}$ and $\phi(3) = \{e, s\}$, in connection with two triangles:

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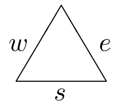
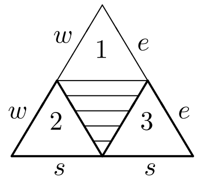
$\Delta(0) \equiv$ 

$\Delta(1) \equiv$ 

We define I_i and J_i as follows: $I_i = J_i$ $(i = 1,2,3,4,5,6,7)$, $I_1 = \{w\}$, $I_2 = \{e\}$, $I_3 = \{s\}$, $I_4 = \{w,e\}$, $I_5 = \{w,s\}$, $I_6 = \{e,s\}$, $I_7 = \{w,e,s\}$.

NOTATION 1. We define a matrix $M\big(\triangle(1)\big) = (m_{ij})$ by

$$m_i^{\,j} = \big|\{s \in S : \tau(s)^{\cup}\big(I_i{}^{\cap}\phi(s)\big) = J_j\}\big|.$$

(We also use $m_I^{\,J}$ and $m_{I_i}^{\,J_j}$ instead of $m_i^{\,j}$ in case no confusion is possible). We can see that $m_1^{\,1} = 1$ and $m_1{}^2 = 0$.

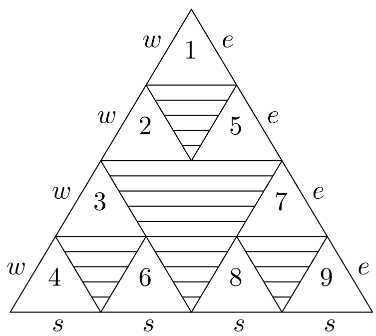
We can obtain $M\big(\triangle(1)\big) = (m_i^{\,j})$ as follows:

$$M\big(\triangle(1)\big) = (m_i^{\,j}) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix},$$

$(J_1\ J_2\ J_3\ J_4\ J_5\ J_6\ J_7)$ is referring columns of the above matrix.

(2) We define a set E as $E = \{w, s, e\}$ and a set $S = \{1,2,3,4,5,6,7,8,9\}$.

$\triangle(2)$ is the symbol of the right picture:



We define τ as follows: $\tau(1) = \{s\}$, $\tau(2) = \{s,e\}$, $\tau(3) = \{s,e\}$, $\tau(4) = \{e\}$, $\tau(5) = \{w,s\}$, $\tau(6) = \{w,e\}$, $\tau(7) = \{w,s\}$, $\tau(8) = \{w,e\}$, $\tau(9) = \{w\}$.

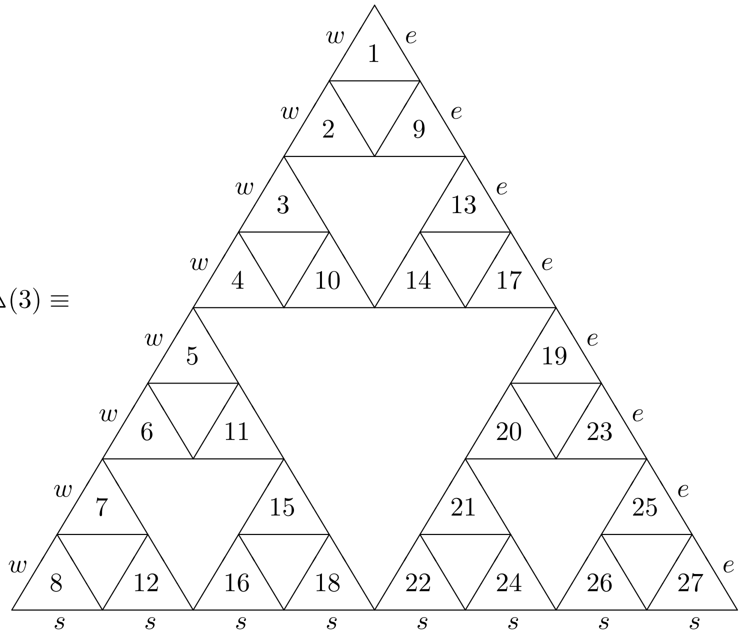
Definition 1. *Let F be a subset of E. We define $F^{\perp} = \{t \in E : t \notin F\}$ and we call it the complementary subset of F for E. We may write $F \oplus F^{\perp} = E$.*

We define a mapping ϕ by $\phi(t) = \{\tau(t)\}^{\perp}$. I_i and J_i are defined as in (1). We use $m_i{}^j$ defined as before and we can see the following the transfer matrix $M\big(\triangle(2)\big)$:

$$M\big(\triangle(2)\big)=\begin{bmatrix} 1&0&0&3&3&0&2\\ 0&1&0&3&0&3&2\\ 0&0&1&0&3&3&2\\ 0&0&0&4&0&0&5\\ 0&0&0&0&4&0&5\\ 0&0&0&0&0&4&5\\ 0&0&0&0&0&0&9 \end{bmatrix}.$$

(3) We consider $\triangle(3)$:

$\triangle(3) \equiv$



(The picture of $\triangle(3)$ may be called the 3rd step figure of Sierpinski triangle or gasketπ).

We define E as before and define $S=\{1,2,3,\ldots,27\}$. I_i and J_i are defined as in (1). We define τ as follows:

$\tau(1)=\{s\}$, $\tau(2)=\tau(3)=\tau(4)=\tau(5)=\tau(6)=\tau(7)=\{e,s\}$, $\tau(8)=\{e\}$,

$\tau(9)=\tau(13)=\tau(17)=\tau(19)=\tau(23)=\tau(25)=\{w,s\}$,

$\tau(10)=\tau(11)=\tau(20)=\tau(21)=\tau(14)=\tau(16)=\{w,e,s\}=E$,

$\tau(12)=\tau(16)=\tau(18)=\tau(22)=\tau(24)=\tau(26)=\{w,e\}$, $\tau(27)=\{w\}$.

We define $\varphi(t)$ as $\{\tau(t)\}^{\perp}$. In this case ${m_i}^j$ is given by

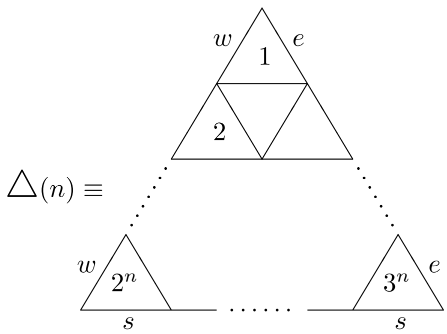
$${m_i}^j=|\{s\in S:\ \tau(s)\cup\big(I_i\cap\varphi(s)\big)=J_j\}|.$$

For $I_1=\{w\}=J_1$, we can see that the only choice is $\tau(27)=\{w\}$ and ${m_1}^1=1$. We can see that J_1 and I_i $(i\neq 1)$ make ${m_i}^1=0$. This way we may obtain the following:

$$M\big(\triangle(3)\big) = \begin{bmatrix} 1 & 0 & 0 & 7 & 7 & 0 & 12 \\ 0 & 1 & 0 & 7 & 0 & 7 & 12 \\ 0 & 0 & 1 & 0 & 7 & 7 & 12 \\ 0 & 0 & 0 & 8 & 0 & 0 & 19 \\ 0 & 0 & 0 & 0 & 8 & 0 & 19 \\ 0 & 0 & 0 & 0 & 0 & 8 & 19 \\ 0 & 0 & 0 & 0 & 0 & 0 & 27 \end{bmatrix},$$

3. A PROPOSITION

In this section, we prove a proposition for Sierpinski triangle or gasket. Let $\triangle(n)$ be the n-step figure of Sierpinski triangle. Let $E = \{w, e, s\}$ be a set of three elements referring $\triangle(0)$. We let $S = \{1, 2, 3, \ldots, 3^n\}$ and define $\tau(t)$ by the usual way for $\triangle(n)$. We define $\varphi(t)$ by $\varphi(t) = \{\varphi(t)\}^{-1}$. We define $I_i = J_i$ as we defined in the section 2. We use $m_i{}^j$ which is defined in the section 2.



Proposition 1. *The n-th step figure of Sierpinski triangle $\triangle(n)$ has the following transfer matrix $M\big(\triangle(n)\big) = (m_i{}^j)$ of $\triangle(n)$:*

$$M\big(\triangle(n)\big) = \begin{bmatrix} 1 & 0 & 0 & 2^n-1 & 2^n-1 & 0 & 3^n-2^{n+1}+1 \\ 0 & 1 & 0 & 2^n-1 & 0 & 2^n-1 & 3^n-2^{n+1}+1 \\ 0 & 0 & 1 & 0 & 2^n-1 & 2^n-1 & 3^n-2^{n+1}+1 \\ 0 & 0 & 0 & 2^n & 0 & 0 & 3^n-2^n \\ 0 & 0 & 0 & 0 & 2^n & 0 & 3^n-2^n \\ 0 & 0 & 0 & 0 & 0 & 2^n & 3^n-2^n \\ 0 & 0 & 0 & 0 & 0 & 0 & 3^n \end{bmatrix}.$$

Proof. (i) We apply the method used in the chapter 2 and obtain $(1\ 0\ 0\ 0\ 0\ 0\ 0)^T$ as the first column of the matrix, where T denotes the symbol of transpose. Similarly, we can prove that the second column of the matrix is $(0\ 1\ 0\ 0\ 0\ 0\ 0)^T$ and the third column of the matrix is $(0\ 0\ 1\ 0\ 0\ 0\ 0)^T$.

(ii) We consider $m_4{}^4$. If $\triangle(3)$ is the case ($n = 3$), we know that $m_4{}^4 = 2^3 = 8$ by the example and hence it is justified for $n = 3$. If $n = 4$, there exist 2^4 triangles each of which has s mark and $\tau(t) \cup \big(\{w, e\} \cap \varphi(t)\big) = \{w, e\}$ gives $m_4{}^4 = 2^4 = 16$. We use a symbol $S_I{}^J$ defined as $S_I{}^J = \{t \in S : \tau(t) \cup \big(I \cap \varphi(t)\big) = J\}$. When $I = I_i$ and $J = J_j$, then $S_I{}^J = S_{I_i}{}^{J_j} = S_i{}^j$ will be used.

If $n = k > 3$, then the number of triangles each of which has s mark is equal to 2^k and hence ${S_I}^I$ gives ${m_4}^4 = 2^k$, where $I = \{w, e\}$. Similarly, we obtain ${m_5}^5 = {m_6}^6 = 2^k$.

(iii) We consider ${m_1}^4$. In the case ${S_I}^I$ gives ${m_7}^7 = 3^k$ $(n = k)$ because each triangle contributes 1 for ${m_7}^7$, where $I = E$. (Each triangle means that a triangle with a number $t \in \{1, 2, 3, \ldots, 3^k\}$.)

(iv) We consider ${m_1}^4$. We take ${S_I}^J$ and triangles with s marks, where $I = \{w\}$ and $J = \{w, e\}$. The number of all triangles with s marks is equal to 2^n for $\triangle(n)$.

We need a symbol $\mathrm{tri}\,(u)$ as the triangle numbered u. Then we see that $\mathrm{tri}\,(2^n)$, $\mathrm{tri}\,(2^n + 2^{n-1})$, $\mathrm{tri}\,(2^n + 2\,(2^{n-1}) + 2^{n-2})$, $\mathrm{tri}\,(2^n + 2\,(2^{n-1}) + 2^{n-2} + 2^{n-1})$, $\mathrm{tri}\,(2^n + 2\,(2^{n-1}) + 2^{n-2} + 2^{n-1} + 2\,(2^{n-2}))$, $\ldots$, $\mathrm{tri}\,(3^n - 1)$ are triangles such that each $\mathrm{tri}\,(u)$ has s mark and $u \in {S_I}^J$. We note that $3^n \notin {S_I}^J$. Thus we obtain ${m_1}^4 = 2^n - 1$. Similarly, we have that ${m_1}^5 = 2^n - 1$.

For ${m_1}^6$, we have $I_1 = \{w\}$ and $J_6 = \{e, s\}$. By ${S_I}^J$ with $I = I_1$ and J_6, we clearly obtain that ${m_1}^6 = 0$.

(v) For ${m_1}^7$, we let $I = \{w\}$ and $J = \{w, e, s\}$.

We know that the total number of $\mathrm{tri}\,(u)$, $u = 1, 2, \ldots, 3^n$, is equal to 3^n. If $I_1 = \{w\}$ is fixed and J_i varies $(i = 1, 2, \ldots, 7)$, then we obtain that ${m_1}^1 + {m_1}^2 + \cdots + {m_1}^7 = 3^n = \sum_{i=1}^{7} {m_1}^i$ and ${m_1}^7 = 3^n - 1 - 2\,(2^n - 1) = 3^n - 2^{n+1} + 1$.

(vi) Consider ${m_4}^7$. We know that ${m_4}^5 = {m_4}^6 = 0$ and ${m_4}^1 = {m_4}^2 = {m_4}^3 = 0$. We also know that ${m_4}^4 = 2^n$ and $\sum_{i=1}^{7} {m_4}^i = 3^n$. Thus we obtain that ${m_4}^7 = 3^n - 2^n$.

The rest is clear and we have proved the proposition.

N-DIMENSIONAL SIERPINSKI TETRAHEDRON

In this section, we take n-dimensional SIERPINSKI tetrahedrons $\triangle_n$, and have Proposition 2 and Theorem 1 about $\triangle_n$, $n \geq 3$.

Definition 2. (i) *Let $E = \{w_1, w_2, \ldots, w_k\}$ be a set of k elements. We let $I_i = \{w_i\}$ $(i = 1, 2, \ldots, k)$, $I_{k+j} = \{w_1, w_{j+1}\}$ $(j = 1, 2, \ldots, k-1)$, $I_{2k} = \{w_2, w_3\}, \ldots, I_\pi = \{w_{k-1}, w_k\}$ $(\pi = 1 + 2 + 3 + \cdots + k - 1)$, $I_{\pi+1} = \{w_1, w_2, w_3\}$, and so on.*

(ii) *$\triangle_n(m)$ is used as the symbol of the k-th step figure of n-dimensional Sierpinski tetrahedron. Let $S = \{1, 2, \ldots, (n+1)^k\}$ for $\triangle_n(m)$.*

(iii) *Let $E = \{w_1, w_2, \ldots, w_{n+1}\}$ for $\triangle_n(m)$.*

(iv) *If $\tau(t)$ and $\varphi(t)$ are defined for $\triangle_{n}(m)$, then the the transfer matrix $M\big(\triangle_{n}(m)\big) = M = (m_i{}^j)$ will be called the combinatorial transfer matrix of the n-dimensional Sierpinski tetrahedron, or $\triangle_{n}(m)$.*

Definition 3. *Let $M\big(\triangle_{n}(m)\big) = M = (m_i{}^j)$ be the combinatorial transfer matrix of $\triangle_{n}(m)$.*

(i) *The submatrix of M for I_i $(i = 1, 2, \ldots, n+1)$ will be denoted by M_{11}.*

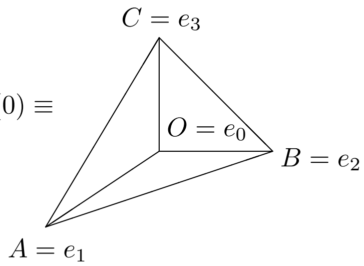
(ii) *The submatrix of M for I_{k+j} will be denoted by M_{22}.*

(iii) *Similarly, we define submatrix M_{ii} $(i = 1, 2, \ldots, n+1)$.*

(iv) *We hence can define M_{ij} as a submatrix of the matrix, for $i, j = 1, 2, \ldots, n+1$.*

We consider the following tetrahedron $\triangle_{3}(0)$ with vertices $A = e_1 = (1\ 0\ 0)$, $B = e_2 = (0\ 1\ 0)$, $C = e_3 = (0\ 0\ 1)$ and $O = e_0 = (0\ 0\ 0)$ in the 3-dimensional Euclidean space $\mathbf{R}^3$:

$\triangle_{3}(0) \equiv$



As in the section 3, we say that the k-step figure of (3-dimensional) Sierpinski tetrahedron and it will be denoted by $\triangle_{3}(k)$. We can define $E, S, \tau(t)$ and $\varphi(t)$ as before.

Definition 4. $\{t \in S : \tau(t) \cup \big(I_i \cap \varphi(t)\big) = J_j\} = S_i{}^j$ *will be called the $S_i{}^j$ set $\big(\triangle_{3}(k)\big)$.*

We know that $|S_i{}^j| = m_i{}^j$. We state the proposition 2.

Proposition 2. (i) *Let $\triangle_{3}(1)$ be the first step of Sierpinski tetraedron. Then the combinatorial transfer matrix $M\big(\triangle_{3}(1)\big) = M = (m_i{}^j)$ of $\triangle_{3}(1)$ is given by the* *following*

$$M = \begin{bmatrix} 1&0&0&0&1&1&1&0&0&0&0&0&0&0&0\\ 0&1&0&0&1&0&0&1&1&0&0&0&0&0&0\\ 0&0&1&0&0&1&0&1&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&1&0&1&1&0&0&0&0&0\\ 0&0&0&0&2&0&0&0&0&0&1&1&0&0&0\\ 0&0&0&0&0&2&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&2&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&2&0&0&1&0&0&1&0\\ 0&0&0&0&0&0&0&0&2&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&0&2&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&0&0&3&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&3&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&3&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&3&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&4 \end{bmatrix}$$

$$= \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14}\\ M_{21} & M_{22} & M_{23} & M_{24}\\ M_{31} & M_{32} & M_{33} & M_{34}\\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix}.$$

(ii) *Let* $\triangle_3(k)$ *be the* k*-th step of Sierpinski tetrahedron. Then the combinartorial transfer matrix* $M\big(\triangle_3(k)\big) = M(3,k)$ *takes the following form:*

$$M(3,k) = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14}\\ M_{21} & M_{22} & M_{23} & M_{24}\\ M_{31} & M_{32} & M_{33} & M_{34}\\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix},$$

(ii)–(1) M_{ii} *is the diagonal matrix* $= \mathrm{diag}(i^k, i^k, \ldots, i^k)$, $(i = 1,2,3,4)$.

(ii)–(2) $M_{ij} = O$ *for* $i > j$, *where* O *denotes the zero matrix.*

(ii)–(3)

$$M_{12} = \begin{bmatrix} \lambda&\lambda&\lambda&0&0&0\\ \lambda&0&0&\lambda&\lambda&0\\ 0&\lambda&0&\lambda&0&\lambda\\ 0&0&\lambda&0&\lambda&\lambda \end{bmatrix},$$

where $\lambda = 2^k - 1$.

(ii)–(4)

$$M_{13} = \begin{bmatrix} \lambda&\lambda&\lambda&0\\ \lambda&\lambda&0&\lambda\\ \lambda&0&\lambda&\lambda\\ 0&\lambda&\lambda&\lambda \end{bmatrix},$$

where $\lambda = 3^k - 2^{k+1} + 1$.

(ii)–(5) $M_{14} = [\lambda \ \ \lambda \ \ \lambda \ \ \lambda]^T$, *where* $\lambda = 4^k - 3^{k+1} + 3 \cdot 2^k - 1$.

(ii)–(6)

$$M_{23} = \begin{bmatrix} \lambda & \lambda & 0 & 0 \\ \lambda & 0 & \lambda & 0 \\ 0 & \lambda & \lambda & 0 \\ \lambda & 0 & 0 & \lambda \\ 0 & \lambda & 0 & \lambda \\ 0 & 0 & \lambda & \lambda \end{bmatrix},$$

where $\lambda = 3^k - 2^k$.

(ii)–(7) $M_{24} = [\lambda \ \ \lambda \ \ \lambda \ \ \lambda \ \ \lambda \ \ \lambda]^T$, *where* $\lambda = 4^k - 2 \cdot 3^k + 2^k$.

(ii)–(8) $M_{34} = [\lambda \ \ \lambda \ \ \lambda \ \ \lambda]^T$, *where* $\lambda = 4^k - 3^k$.

The proof of Proposition 2 is similar to the proof of Proposition 1 and we omit the proof of the proposition.

In the n-dimensional Euclidean space $\mathbf{R}^n$, a symbol $\triangle_n$ denotes the n-dimensional tetrahedron with vertices $e_0 = (0 \ \ 0 \ \ldots \ 0)$, $e_1 = (1 \ \ 0 \ \ 0 \ \ldots \ 0)$, $e_2 = (0 \ \ 1 \ \ 0 \ \ldots \ 0), \ldots, e_n = (0 \ \ 0 \ \ldots \ 0 \ \ 1)$. We may use $\triangle_n(k)$ as a symbol for the k-th step figure of an n-dimensional SIERPINSKI tetrahedron. We define a set E and state Theorem 1.

Definition 5. *Let* $E = \{w_1, w_2, \ldots, w_{n+1}\}$. $\langle \dot{e}_1 \ \ e_2 \ \ e_3 \ldots e_n \ \ e_0 \rangle$ *denotes the* $(n-1)$ *dimensional tetrahedron formed by* $e_2, e_3, \ldots, e_n$ *and* e_0, *where* $\dot{e}_1$ *means that* e_1 *is missing.*

We write w_1 as $w_1 = \langle \dot{e}_1 \ \ e_2 \ldots e_n \ \ e_0 \rangle$ and $w_2 = \langle e_1 \ \ \dot{e}_2 \ \ e_3 \ldots e_n \ \ e_0 \rangle$. Similarly we can write $w_3, w_4, \ldots, w_{n+1}$.

(We have used $E = \{w, e, s\}$ in the section 2 and we may rewrite as $w_1 = w$, $w_2 = e$ and $w_3 = s$.)

We state theorem 1.

Theorem 1. *A combinatorial transfer matrix* $M\left(\triangle_n(k)\right) = M = (M_{ij})$ *takes the following*:

(1) $M_{ii} - \mathrm{diag}\,(i^k \ \ i^k \ \cdots \ i^k)$ *is a diagonal matrix* $(i = 1, 2, \ldots, n+1)$ *and a* $\binom{n+1}{i}$ *by* $\binom{n+1}{i}$ *matrix.*

(2) $M_{ij} = O$ *is the zero matrix if* $j > i$.

(3) *For each* $i \in \{1, 2, \ldots, 2^{n+1} - 1\}$, $\sum_{j=1}^{n+1} m_i{}^j = (n+1)^k$.

(4) *If* u_1 *and* u_2 *are non-zero elements of* M_{ij}, *then* $u_1 = u_2$.

(5) *Suppose that* m_1, m_2 *and* m_3 *are non-zero elements of* M_{ii}, $M_{i\ i+1}$ *and* $M_{i+1\ i+1}$, *respectively, then* $m_1 + m_2 = m_3$, $(i = 1, 2, \dots, n+1)$.

(6) *Suppose that* m_1, m_2 *and* m_3 *are non-zero elements of* $M_{i\ i+1}$, $M_{i\ i+2}$ *and* $M_{i+1\ i+2}$, *respectively, then* $m_1 + m_2 = m_3$.

Proof of Theorem 1. We assume E is defined for $\triangle_n(0)$ by Definition 5. Suppose we have defined $S = \{1, 2, \dots, (n+1)^k\}$ for $\triangle_n(k)$ and hence we can say that tri(u) $(u = 1, 2, \dots, (n+1)^k)$ is defined as we had in the chapter 2–(2), where tri (u) means that a small tetrahedron with the number u. We also have defined $\tau(t)$ for $t \in S$.

(1) We now consider M_{11}. There exists u in S such that $\tau(u) = \{w_1\}$ and $\varphi(u) = \{\tau(u)\}^{\perp}$. Thus ${S_1}^1 = \{u\}$ and ${m_1}^1 = 1$. It is clear that ${m_j}^1 = 0$ $(j = 2, 3, \dots, n+1)$ by ${S_j}^1$.

By the combinatorial observation we conclude that $M_{11} = I$, the identity matrix of rank $(n+1)$.

Consider now M_{22}. Let $I = \{w_1, w_2\} = J$. Assume that $k = 1$. Then there exist u_1 and u_2 in S such that $\tau(u_1) = \{w_1\}$ and $\tau(u_2) = \{w_2\}$. Thus ${m_I}^J = 2$. If $k = 2$, there exist u_3 and u_4 in S such that $\tau(u_3) = \{w_1, w_2\} = \tau(u_4)$. We know that tri (u_1) and tri (u_2) make $\tau(u_1) = \{w_1\}$ and $\tau(u_2) = \{w_2\}$, respectively. Thus we obtain that ${m_I}^J = 2^2$. This way we obtain that ${m_I}^J = 2^p$ when $k = p$. The rest is clear and hence we have $M_{22} = \text{diag}], (2^k\ \ 2^k \dots 2^k) = 2^k \cdot I$, where I is the identity matrix of rank $\binom{n+1}{2}$.

(2) It is clear by ${S_I}^J$.

(3) Let $i \in \{1, 2, \dots, 2^{k+1} - 1\}$. Consider $u \in \{1, 2, \dots, (n+1)^k\} = S$. tri (u) contributes 1 to $\sum_{j=1}^{\pi} {m_i}^j$, where π denotes $\pi = 2^{n+1} - 1$. Therefore we obtain that $\sum_{j=1}^{\pi} {m_i}^j = (n+1)^k$.

(4) By a combinatorial observation or a statistical view the assertion is justified.

(5) We know that $M_{n\ n}$ is a $(n+1)$ by $(n+1)$ matrix and $M_{n+1\ n+1}$ is a number or $M_{n+1\ n+1} = (n+1)^k$. By (3), the assertion is true for this case. If $i = n-1$, the assertion is also true. We omit the rest of the proof of (5).

(6) See Proposition 2 for a special case and we omit the rest of the proof of (6).

Note 1. For $\triangle_3(2)$, we defined S as $\{1, 2, \dots, 4^2\}$. We may redefine S as $S = \{1-1,\ 1-2,\ 1-3,\ 1-4,\ 2-1,\ 2-2,\ 2-3,\ 2-4,\ 3-1,\ 3-2,\ 3-3,\ 3-4,\ 4-1,\ 4-2,\ 4-3,\ 4-4\}$.

5. FRACTAL DIMENSION THEOREM

We consider *Fractal Dimensions* of n-dimensional SIERPINSKI tetrahedron $\triangle_{n}(\pi)$ in connection with a transfer matrix of n-dimensional SIERPINSKI tetrahedron. We start with the following definition.

Definition 6 [**1**, p. 173–174]. *Let (X, d) denote a complete metric space. Let $A \in H(X)$* (see [**1**] for $H(X)$) *be a non-empty compact subset of X. Let $\varepsilon > 0$. Let $B(x, \varepsilon)$ denote the closed ball of radius ε and center at a point $x \in X$. For each $\varepsilon > 0$, let $N(A, \varepsilon)$ denote the smallest number of closed ball $B(x, \varepsilon)$ of radius ε needed to cover A. If $D = \lim_{\varepsilon \to 0} (\ln (N(A, \varepsilon))/\ln (1/\varepsilon))$ exists, then D is called the fractal dimension of A. We use the Euclidean metric d and $X = R$. We will use the notation $D = D(A)$, and will say that A has fractal dimension $D = D(A)$.*

We use the box counting theorem.

Theorem 2 [**1**, p 136]. (The box counting theorem) *Let $A \in H(R^m)$, where the Euclidean metric is used. Cover R^m by closed just-touching square boxed of side length $1/2^n$. Let $N(A, n)$ denote the number of boxed of side length $1/2^n$ which intersect the attractor. If $D = \lim_{n \to \infty} (\ln(N(A, n))/\ln(2^n))$, then A has fractal dimension D.*

Theorem 1 states that $M_{n+1\ n+1} = (n + 1)^k$ for $\triangle_{n}(k)$.

Let $2^{n+1} - 1 = \alpha$. Then ${m_\alpha}^\alpha = (n + 1)^k$ by Theorem 1. If we write $\triangle_{m}(n)$, then ${m_\alpha}^\alpha = (m + 1)^n$. We shall use it in Theorem 3.

Definition 7. We define $\triangle_{m}(\pi)$ as $\triangle_{m}(\pi) = \lim_{n \to \infty} \triangle_{m}(n)$ and we may say that $\triangle_{m}(\pi)$ denotes the symbol of an m-dimensional SIERPINSKY tetrahedron.

Theorem 3. *An m-dimensional Sierpinski tetrahedron $\triangle_{m}(\pi)$ has fractal dimension*

$$D = D\big(\triangle_{m}(\pi)\big) = \lim_{n \to \infty} \frac{\ln\big(N\big(\triangle_{m}(n)\big)\big)}{\ln(2^n)} = \lim_{n \to \infty} \frac{\ln(m + 1)^n}{\ln(2^n)} = \frac{\ln(m + 1)}{\ln 2}\,.$$

We apply the box counting theorem to $\triangle_{m}(n)$ and obtain $N\big(\triangle_{m}(n)\big) = (m + 1)^n$. The rest of the proof is clear.

We refer to [**2**] for a detailed proof of Theorem 3 . (We note that an m-dimensional SIERPINSKI tetrahedron is not usual one.)

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