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A CLASS OF C^∞ SLOWLY VARYING FUNCTIONS

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For a given slowly varying function L we find explicitely another slowly varying function $g = g(L(x))$ such that $g \in C^\infty$ and $g(x) \sim L(x),\ x \to \infty$. A remarkable property of g is possibility of analytic continuation on the right complex half-plane.

1. INTRODUCTION

In this article we treat a subclass of KARAMATA's slowly varying functions (svf) with analytic properties. Such are, for example:

$$\ln^a x,\ \ln^b(\ln x),\quad a, b \in \mathbf{R};\quad \exp(\ln x)^c,\ 0 < c < 1;\quad \exp\left(\frac{\ln x}{\ln\ln x}\right),\ \text{e.t.c.}$$

An excellent survey of characterisation, representation etc., connected with regular variation is given in [**1**] and [**2**]; therefore, we suppose that the reader is familliar with it. In general, despite their name, the behaviour of slowly varying functions shows great irregularities. For example (BINGHAM [**1**], p.16), for svf $L(x) = \exp\left((\ln x)^{1/3} \cos(\ln x)^{1/3}\right)$:

$$\liminf_{x\to\infty} L(x) = 0;\quad \limsup_{x\to\infty} L(x) = +\infty.$$

Moreover, (ADAMOVIĆ [**3**], [**4**]) for any a, b $(a < b)$ from the segment $[0, \infty]$ there is a continuous svf $L(x)$, such that

$$\liminf_{x\to\infty} L(x) = a;\quad \limsup_{x\to\infty} L(x) = b.$$

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Also, other analytical aspects as monotonicity, differentiability etc., are in general, out of question.

Anyway, since the definition of svfs includes asymptotic relations near infinity, their behaviour on finite segments are of secondary importance; therefore, we could always "fix" them in a way to be locally bounded and $O(1)$ when $x \to 0^+$.

The class of such svfs is called $\operatorname{Loc}(L)$. It is evident that their asymptotic properties at infinity are not disturbed.

From the other side, theorems as:

A. (De Bruijn, 1959): *For every svf $L(x)$ there exist another svf $L_1(x) \in C^\infty$ such that $L_1(x) \sim L(x),\ \ x \to \infty$;* or

B. (Adamovic, 1966): *For every svf $L(x)$ and arbitrary monotone increasing, unbounded sequence (x_n), there exist svf $L_0(x) \in C^\infty$ such that $L_0(x) \sim L(x),\ \ x \to \infty$ and $L_0(x_n) = L(x_n)$ for all large n;*

shows that, for sufficiently large values of variable, "bad" svfs could be replaced by the analytic ones with desirable properties.

The only problem with analytic svfs L_0 and L_1 in cited theorems is that due to their construction (see [**1**], p.14, 15), they can not be applied to concrete problems.

Hence, our task in this article is to find out explicitly, for a given svf $L(x)$, another svf $g = g\big(L(x)\big)$ such that $g \in C^\infty$ and $g(x) \sim L(x),\ \ x \to \infty$.

A remarkable property of our svf g is the possibility of analytic continuation on the right complex half-plane without loosing the regularity mode, i.e.

$$g(xe^{i\phi}) \sim g(x) \sim L(x),\ \ x \to \infty,\ |\phi| < \pi/2.$$

2. RESULTS

Proposition 1. *For any svf $L \in \operatorname{Loc}(L)$, define L_* as*

$$L_*(z) := z \int_0^\infty e^{-zt} L(t)\,\mathrm{d}t,\ \ z \in \mathbf{Z},\ \ |\arg z| < \pi/2.$$

Then

$$L_*(z) \sim L_*(|z|) \sim L(1/|\,z\,|),\ \ |z| \to 0^+,\ \ |\arg z| < \pi/2.$$

More precisely

$$\operatorname{Re} L_*(z) \sim L(1/|z|);\ \ \operatorname{Im} L_*(z) = o(L(1/|z|);\ \ |z| \to 0^+,\ \ |\arg z| < \pi/2.$$

This proposition shows that $L_*(x), x \in \mathbf{R}^+$ is an C^∞ svf (as a Laplace transform of a positive function L) and $L_*(z)$ is it's analytic continuation on the right complex half-plane.

For the proof we need a well-known result:

Lemma 1. ([**1**] p.199, ALJANCIC) *If for some* $\delta > 0$, $\int\limits_0^\infty t^\eta |k(t)|\,\mathrm{d}t$ *is convergent for* $-\delta \le \eta \le \delta$ *and* $L \in \mathrm{Loc}\,(L)$, *then*

$$\int\limits_0^\infty k(t)L(xt)\,\mathrm{d}t \sim L(x)\int\limits_0^\infty k(t)\,\mathrm{d}t, \quad x \to \infty.$$

If $\int\limits_0^\infty k(t)\,\mathrm{d}t = 0$, *then*

$$\int\limits_0^\infty k(t)L(xt)\,\mathrm{d}t = o\big(L(x)\big), \quad x \to \infty.$$

We shall prove the second part of the Proposition 1.

Proof. Let $z = re^{i\phi}$, $|z| = r$, $\arg z = \phi \in (-\pi/2, \pi/2)$. Since

$$L_*(z) = L_*(re^{i\phi}) = re^{i\phi}\int\limits_0^\infty e^{-tre^{i\phi}}L(t)\,\mathrm{d}t = r\int\limits_0^\infty e^{-tr\cos\phi}\cdot e^{i(\phi - tr\sin\phi)}L(t)\,\mathrm{d}t,$$

we have

$$\mathrm{Re}\,L_*(z) = r\int\limits_0^\infty e^{-tr\cos\phi}\cos(\phi - tr\sin\phi)L(t)\,\mathrm{d}t;$$
$$\mathrm{Im}\,L_*(z) = r\int\limits_0^\infty e^{-tr\cos\phi}\sin(\phi - tr\sin\phi)L(t)\,\mathrm{d}t.$$

Substituting $tr = u$ and applying Lemma 1 (with $\delta = 1/2$), $1/r \to +\infty$, we get

$$\begin{aligned}\mathrm{Re}\,L_*(z) &= \int\limits_0^\infty e^{-u\cos\phi}\cos(\phi - u\sin\phi)L(u/r)\,\mathrm{d}u\\ &\sim L(1/r)\int\limits_0^\infty e^{-u\cos\phi}\cos(\phi - u\sin\phi)\,\mathrm{d}u, \quad r \to 0^+,\\ \mathrm{Im}\,L_*(z) &= \int\limits_0^\infty e^{-u\cos\phi}\sin(\phi - u\sin\phi)L(u/r)du\\ &\sim L(1/r)\int\limits_0^\infty e^{-u\cos\phi}\sin(\phi - u\sin\phi)\,\mathrm{d}u, \quad r \to 0^+, \ |\phi| < \pi/2.\end{aligned}$$

Integrals on the right exists ($\cos\phi > 0$) and doesn't depend on the parameter ϕ. First is equal to 1 and second to 0, i.e. the Proposition 1 is proved.

In the next theorem we show that the complex-valued function $L_*(z)$ is playing a role of a svf on the right complex half-plane.

Proposition 2. *For any $c \in \mathbf{Z}^+$ from right complex half-plane,*

$$\frac{L_*(cz)}{L_*(z)} \to 1, \ \ |z| \to 0^+, \ \ |\arg z| < \pi/2 - |\arg c|.$$

Proof. Since $\arg(cz) = \arg c + \arg z$, follows

$$|\arg(cz)| \le |\arg c| + |\arg z| < \pi/2.$$

Applying the Proposition 1, we get

$$L_*(cz) \sim L(1/|cz|) = L\left(\frac{1}{|c|} \cdot \frac{1}{|z|}\right) \sim L(1/|z|) \sim L_*(z), \ \ |z| \to 0^+.$$

Now we could define a complex regularly varying function (rvf) $R(z)$ of index α in the usual manner, as

$$\text{(A)} \qquad R(z) := (1/z)^\alpha L_*(z), \ \ |\arg z| < \pi/2, \ \ \alpha \in \mathbf{R}$$

(in definition of z^α we allways take the branch which is positive for positive z). Then

$$\frac{R(z)}{R(|z|)} \sim e^{i\alpha \arg z}, \ \ |z| \to 0^+.$$

An alternative way is given in the next Proposition.

Proposition 3. *Let*

$$\text{(B)} \qquad R_*^{(\alpha)}(z) := \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zt} t^{\alpha-1} L(t)\,\mathrm{d}t, \ \ \alpha > 0, \ \ |\arg z| < \pi/2.$$

Then

$$R_*^{(\alpha)}(z) \sim (1/z)^\alpha L_*(z); \ \ |z| \to 0^+,$$

i.e. $R_*^{(\alpha)}(z)$ *is regularly varying with index* $\alpha > 0$.

Proof. We have $(z = re^{i\phi})$:

$$\operatorname{Re} R_*^{(\alpha)}(z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-tr\cos\phi} \cos(tr\sin\phi)\, t^{\alpha-1} L(t)\,\mathrm{d}t,$$

$$\operatorname{Im} R_*^{(\alpha)}(z) = -\frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-tr\cos\phi} \sin(tr\sin\phi) t^{\alpha-1} L(t)\,\mathrm{d}t.$$

Substituting $tr = u$ and applying Lemma 1 (which is valid for $\delta = \alpha/2$), we get

$$\operatorname{Re} R_*^{(\alpha)}(z) \sim \frac{L(1/r)}{\Gamma(\alpha)} \cdot \frac{1}{r^\alpha} \int_0^\infty e^{-u\cos\phi} \cos(u\sin\phi)\, u^{\alpha-1}\mathrm{d}u,$$

$$\operatorname{Im} R_*^{(\alpha)}(z) \sim -\frac{L(1/r)}{\Gamma(\alpha)} \cdot \frac{1}{r^\alpha} \int_0^\infty e^{-u\cos\phi} \sin(u\sin\phi) u^{\alpha-1}\mathrm{d}u,$$

when $r \to 0^+$.

For the evaluation of integrals on the right side, we use (see [**5**], p.144):

$$\int_0^\infty e^{-s^a \cos ab} \cos(s^a \sin ab)\,\mathrm{d}s = \frac{\cos b}{a}\,\Gamma(1/a),$$
$$\int_0^\infty e^{-s^a \cos ab} \sin(s^a \sin ab)\,\mathrm{d}s = \frac{\sin b}{a}\,\Gamma(1/a), \quad a > 0,\ |ab| < \pi/2.$$

Substituting $s = u^{1/a},\ \alpha = 1/a,\ b = \alpha\phi$, we get

$$\int_0^\infty e^{-u\cos\phi} u^{\alpha-1} \cos(u \sin\phi)\,\mathrm{d}u = \Gamma(\alpha)\cos\alpha\phi,$$
$$\int_0^\infty e^{-u\cos\phi} u^{\alpha-1} \sin(u \sin\phi)\,\mathrm{d}u = \Gamma(\alpha)\sin\alpha\phi, \quad \alpha > 0,\ |\phi| < \pi/2,$$

i.e.

$$\begin{aligned} R_*^{(\alpha)}(z) &= \operatorname{Re} R_*^{(\alpha)}(z) + i \operatorname{Im} R_*^{(\alpha)}(z) \\ &\sim \frac{L(1/r)}{r^\alpha}\,(\cos\alpha\phi - i\sin\alpha\phi) \sim (1/z)^\alpha L_*(z),\ \ |z| \to 0^+,\ \ |\arg z| < \pi/2. \end{aligned}$$

Now we give an asymptotic estimation of the n-th derivative rvf of $R_*^{(\alpha)}(z)$, which itself represents an analogy with well-known class SR_α (smoothly varying functions, [**1**], p.44).

Proposition 4.

$$\frac{z^n\,\big(R_*^{(\alpha)}(z)\big)^{(n)}}{R_*^{(\alpha)}(z)} \sim (-1)^n \alpha(\alpha+1)\cdots(\alpha+n-1);\ \ |z| \to 0^+,\ \ \alpha > 0,\ n \in \mathbf{N}.$$

Proof. This is a simple consequence of the previous proposition, i.e.

$$\begin{aligned} \big(R_*^{(\alpha)}(z)\big)^{(n)} &= \frac{(-1)^n}{\Gamma(\alpha)} \int_0^\infty e^{-zt} t^{\alpha-1+n} L(t)\,\mathrm{d}t \\ &\sim (-1)^n\,\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \cdot \frac{L_*(z)}{z^{\alpha+n}};\ \ |z| \to 0^+,\ |\arg z| < \pi/2,\ \alpha > 0. \end{aligned}$$

An alternative to $L_*(z)$ is svf $L^*(z)$ defined as

$$L^*(z) := z \int_0^\infty e^{-zt} L(1/t)\,\mathrm{d}t, \quad L \in \operatorname{Loc}(L).$$

In the same way, we have

$$L^*(z) \in C^\infty;\ \ L^*(z) \sim L(|z|),\ |z| \to \infty.$$

All propositions 1–4 are also valid for $L^*(z)$ with 0^+ and ∞ reversed.

Proposition 5. *If* $a(s) \to \infty,\ s \to \infty;\ a(s) \sim b(s),\ s \to \infty,$ *then*

$$L^*\big(a(s)\big) \sim L^*\big(b(s)\big),\quad s \to \infty.$$

Proof. From the assumed condition it follows

$$a(s) = b(s)\big(1 + o(1)\big),\quad s \to \infty.$$

Therefore, for large enough $s > s_0$, we could find $\varepsilon = \varepsilon(s_0)$ such that

$$b(s)(1-\varepsilon) \le a(s) \le b(s)(1+\varepsilon),\ \ s > s_0.$$

Now

$$\begin{aligned} L^*\big(a(s)\big) &= a(s)\int_0^\infty e^{-a(s)t}L(1/t)\,\mathrm{d}t \le b(s)(1+\varepsilon)\int_0^\infty e^{-b(s)(1-\varepsilon)t}L(1/t)\,\mathrm{d}t \\ &= \frac{b(s)(1+\varepsilon)}{b(s)(1-\varepsilon)} \cdot L^*\big(b(s)(1-\varepsilon)\big), \end{aligned}$$

and, analogously:

$$L^*\big(a(s)\big) \ge \frac{1-\varepsilon}{1+\varepsilon}\, L^*\big(b(s)(1+\varepsilon)\big),\quad s > s_0.$$

Hence, for fixed ε we have

$$\limsup_{s\to\infty} \frac{L^*\big(a(s)\big)}{L^*\big(b(s)\big)} \le \frac{1+\varepsilon}{1-\varepsilon} \lim_{s\to\infty} \frac{L^*\big(b(s)(1-\varepsilon)\big)}{L^*\big(b(s)\big)} = \frac{1+\varepsilon}{1-\varepsilon},$$

and

$$\liminf_{s\to\infty} \frac{L^*\big(a(s)\big)}{L^*\big(b(s)\big)} \ge \frac{1-\varepsilon}{1+\varepsilon} \lim_{s\to\infty} \frac{L^*\big(b(s)(1+\varepsilon)\big)}{L^*\big(b(s)\big)} = \frac{1-\varepsilon}{1+\varepsilon}.$$

Therefore

$$\frac{1-\varepsilon}{1+\varepsilon} \le \liminf_{s\to\infty} \frac{L^*\big(a(s)\big)}{L^*\big(b(s)\big)} \le \lim_{s\to\infty} \frac{L^*\big(a(s)\big)}{L^*\big(b(s)\big)} \le \limsup_{s\to\infty} \frac{L^*\big(a(s)\big)}{L^*\big(b(s)\big)} \le \frac{1+\varepsilon}{1-\varepsilon}.$$

Since ε is arbitrary small, the conclusion follows.

We could define a regularly varying function R^*_α of index α in the usual way as

$$R^*_\alpha(x) := x^\alpha L^*(x),\quad \alpha \in \mathbf{R}.$$

We have

$$R^*_\alpha(x) \in C^\infty;\ R^*_\alpha(x) \sim x^\alpha L(x),\quad x \to \infty,\ \ L \in \mathrm{Loc}\,(L).$$

Now we prove that R^*_α is smoothly varying in the sense defined in ([**1**], p.44).

Proposition 6.

$$\frac{x^n\big(R^*_\alpha(x)\big)^{(n)}}{R^*_\alpha(x)} \sim \alpha(\alpha-1)\cdots(\alpha-n+1), \quad x\to\infty.$$

Proof. Since

$$\left(\frac{L^*(x)}{x}\right)^{(n)} = \int_0^\infty (-1)^n t^n e^{-xt} L(1/t)\,\mathrm{d}t, \ n\in\mathbf{N},$$

we get

$$x^n\big(R^*_\alpha(x)\big)^{(n)} = x^n\left(x^{\alpha+1}\left(\frac{L^*(x)}{x}\right)\right)^{(n)} = x^n\sum_{k=0}^{n}(x^{\alpha+1})^{(k)}\left(\frac{L^*(x)}{x}\right)^{(n-k)}\binom{n}{k}$$

$$= x^{\alpha+1}\int_0^\infty e^{-xt}\left(\sum_{k=0}^{n}\binom{n}{k}(\alpha+1)\alpha(\alpha-1)\cdots(\alpha+2-k)(-xt)^{n-k}\right)L(1/t)\,\mathrm{d}t.$$

Substituting $xt=u$ and applying Lemma 1, we obtain

$$x^n\big(R^*_\alpha(x)\big)^{(n)}$$
$$\sim(-1)^n n!\,x^\alpha L(x)\cdot\left(1+\sum_{k=1}^{n}(-1)^k\,\frac{(\alpha+1)\alpha(\alpha-1)\cdots(\alpha+2-k)}{k!}\right), \quad x\to\infty.$$

The expression in parentheses is equal to $(-1)^n\,\dfrac{\alpha(\alpha-1)\cdots(\alpha-1+n)}{n!}$

Since $x^\alpha L(x)\sim R^*_\alpha(x)$, $x\to\infty$, validity of the statement follows.

As a consequence we get the following

Proposition 7. $\big(R^*_\alpha(x)\big)^{(n)} = O\left(\frac{R^*_\alpha(x)}{x^n}\right)$, *where absolute constant in O does not depend on x.*

Even more important consequence is that our rvf R^*_α is an explicit realisation of a function g from the statement ([**1**], p.45):

Smooth variation theorem. *For every rvf f there is smoothly varying function g such that*

$$g(x)\sim f(x), \quad x\to\infty.$$

For an illustration, we apply our concept of C^∞ svf to investigate the asymptotic behaviour of

$$\sum_{k\le n} c^*_k\binom{n}{k}z^k, \quad k\in\mathbf{N}, \ z\in\mathbf{Z},$$

where c_k^* is a regularly varying sequence generated by rvf $R_\alpha^*(x)$, i.e

$$c_k^* := k^a L^*(k) = k^a L_k^*, \quad a \le 0, \; k \in \mathbf{N},$$
$$c_n^* \sim n^a L(n) = c_n, \quad n \to \infty,$$

for any svf L (here $L \in \operatorname{Loc} L$ is self-evident).

We shall show that on different sets of complex variable z there is different asymptotics of the given sum.

Lemma. *The sequence (c_k^*) of index $-(a+1)$ have the following integral representation:*

$$c_k^* := \frac{L_k^*}{k^{a+1}} = \int_0^\infty e^{-kt} u(a,t)\,\mathrm{d}t,$$

where $u(a,t)$ is given by

$$u(a,t) = \begin{cases} 1/\Gamma(a) \int_0^t (t-s)^{a-1} L(1/s)\,\mathrm{d}s & (a>0), \\ L(1/t) & (a=0). \end{cases}$$

Proof. It follows

$$c_k^* = \frac{L_k^*}{k^{a+1}} = \frac{L^*(k)}{k} \cdot \frac{1}{k^a} = \frac{1}{\Gamma(a)} \int_0^\infty e^{-kt} L(1/t)\,\mathrm{d}t \cdot \int_0^\infty e^{-kt} t^{a-1}\,\mathrm{d}t$$
$$= \frac{1}{\Gamma(a)} \int_0^\infty e^{-kt} \left(\int_0^t (t-s)^{a-1} L(1/s)\,\mathrm{d}s \right) \mathrm{d}t, \quad a>0,$$

according to the well-known convolution of the Laplace transform.

Now we could formulate **Theorem A.** *For $|z+1|>1, \; z \in \mathbf{Z}, \; a \ge 0:$*

$$\sum_{k \le n} \frac{L_k^*}{k^a} \binom{n}{k} z^k \sim \frac{L_n^*}{n^a} (1+1/z)^a (z+1)^n, \quad n \to \infty.$$

Proof. From the cited Lemma we obtain an integral representation of

$$f_n(a,z) := \sum_{k=0}^n \frac{L^*(k+1)}{(k+1)^{a+1}} \binom{n}{k} z^k = \sum_{k=0}^n \binom{n}{k} z^k \int_0^\infty e^{-(k+1)x} u(a,x)\,\mathrm{d}x$$
$$= \int_0^\infty e^{-x} u(a,x) \sum_{k=0}^n \binom{n}{k} (ze^{-x})^k\,\mathrm{d}x = \int_0^\infty e^{-x} u(a,x)(1+ze^{-x})^n\,\mathrm{d}x.$$

Define $\xi_n := \ln(1+n^{-1/2}), \; n \in \mathbf{N}$. Then

$$f_n(a,z) = \left(\int_0^{\xi_n} + \int_{\xi_n}^\infty \right)(\cdot) = I_1 + I_2.$$

Since $x \mapsto |z+1|e^{-x} + 1 - e^{-x}$ is monotone decreasing for $|z+1| > 1$, we have

$$\begin{aligned}
I_2 &= \int_{\xi_n}^{\infty} e^{-x} u(a,x)(1+ze^{-x})\,\mathrm{d}x = O\Big(\int_{\xi_n}^{\infty} e^{-x} u(a,x)\,|1+ze^{-x}|^n \mathrm{d}x\Big)\\
&= O\Big(\int_{\xi_n}^{\infty} e^{-x} u(a,x)(|z+1|e^{-x} + 1 - e^{-x})^n\,\mathrm{d}x\Big)\\
&= O\Big((|z+1|e^{-\xi_n} + 1 - e^{-\xi_n})^n \int_0^{\infty} e^{-x} u(a,x)\,\mathrm{d}x\Big)\\
&= O\bigg(\Big(|z+1| - \frac{|z+1|-1}{1+\sqrt{n}}\Big)^n\bigg) = O\bigg(|z+1|^n \exp\Big(-\sqrt{n}\,\frac{|z+1|-1}{|z+1|}\Big)\bigg).
\end{aligned}$$

Estimating I_1, we use

$$\ln\Big(1 + \frac{e^x - 1}{z+1}\Big) = \frac{x}{z+1} + O(x^2), \quad x \in (0,\xi_n), \quad |z+1| > 1.$$

Now,

$$\begin{aligned}
I_1 &= \int_0^{\xi_n} e^{-x} u(a,x)\,(1+ze^{-x})^n\,\mathrm{d}x\\
&= (z+1)^n \int_0^{\xi_n} e^{-x} u(a,x) \exp\bigg(n \ln\Big(e^{-x}\Big(1 + \frac{e^x-1}{z+1}\Big)\Big)\bigg)\mathrm{d}x\\
&= (z+1)^n \int_0^{\xi_n} e^{-x} u(a,x) exp\Big(n\Big(-x + \frac{x}{z+1} + O(x^2)\Big)\Big)\mathrm{d}x\\
&= (z+1)^n \int_0^{\xi_n} e^{-x} u(a,x) exp\Big(-\frac{nzx}{z+1}\Big) \exp\big(O(nx^2)\big)\,\mathrm{d}x.
\end{aligned}$$

Since $e^t = 1 + O(te^t),\ t \in (0,+\infty)$ and, for $x \in (0,\xi_n)$

$$nx^2 = O\big(n\xi_n^{\,2}\big) = O\big(n \ln^2(1+n^{-1/2})\big) = O\big(O(1)\big) = O(1),$$

it follows

$$\begin{aligned}
I_1 = (z+1)^n &\int_0^{\xi_n} e^{-x} u(a,x) e^{-nzx/(z+1)}\,\mathrm{d}x\\
&+ n(z+1)^n \int_0^{\xi_n} e^{-x} u(a,x) e^{-nzx/(z+1)} O(x^2)\, e^{O(nx^2)}\,\mathrm{d}x\\
= (z+1)^n &\int_0^{\infty} e^{-x} u(a,x) e^{-nzx/(z+1)}\,\mathrm{d}x - (z+1)^n \int_{\xi_n}^{\infty} e^{-x} u(a,x) e^{-nzx/(z+1)}\,\mathrm{d}x\\
&+ O\Big(n|z+1|^n \int_0^{\xi_n} x^2 e^{-x} u(a,x) e^{-nzx/(z+1)}\,\mathrm{d}x\Big) = I_{12} + I_{13} + I_{14}.
\end{aligned}$$

Since for $|z+1|>1$ we have that $\mathrm{Re}\,\dfrac{z}{z+1}=1-\dfrac{\cos\arg(z+1)}{|z+1|}>0,\ z\in\mathbf{Z}$, applying cited Propositions, we obtain

$$\begin{aligned}
I_{12}&=(z+1)^n\int_0^\infty e^{-(1+nz/(z+1))x}u(a,x)\,dx\\
&=(z+1)^n\,\frac{L^*\big(1+nz/(z+1)\big)}{\big(1+nz/(z+1)\big)^{a+1}}\sim(z+1)^{n+a+1}\,\frac{L^*(n)}{(nz)^{a+1}}\,,\quad n\to\infty;\\
|I_{13}|&=O\Big(|z+1|^n\int_{\xi_n}^\infty e^{-x}u(a,x)\,\exp\Big(-nx\mathrm{Re}\,\frac{z}{z+1}\Big)\,\mathrm{d}x\Big)\\
&=O\Big(|z+1|^n e^{-n\xi_n\mathrm{Re}\,\frac{z}{z+1}}\int_0^\infty e^{-x}u(a,x)\,\mathrm{d}x\Big)\\
&=O\Big(|z+1|^n\,(1+n^{-1/2})^{-n\mathrm{Re}\,\frac{z}{z+1}}\Big)=O\Big(|z+1|^n\,e^{-\sqrt{n}\,\mathrm{Re}\,\frac{z}{z+1}}\Big),
\end{aligned}$$

and, similarly,

$$\begin{aligned}
I_{14}&=O\Big(n|z+1|^n\int_0^{\xi_n}x^2e^{-x}u(a,x)\,\exp\Big(-nx\mathrm{Re}\,\frac{z}{z+1}\Big)\,\mathrm{d}x\Big)\\
&=O\Big(n|z+1|^n\int_0^\infty x^2e^{-(1+n\mathrm{Re}\,\frac{z}{z+1})\,x}\,\mathrm{d}x\Big)\\
&=O\Big(n|z+1|\,\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Big(\frac{L^*(s)}{s^{a+1}}\Big)\Big)_{[s=1+n\mathrm{Re}\,\frac{z}{z+1}]}\qquad\text{(according to the Proposition 7)}\\
&=O\Big(n|z+1|^n\,\frac{L^*(s)}{s^{a+3}}\Big)_{[s=1+n\mathrm{Re}\,\frac{z}{z+1}]}=O\Big(|z+1|^n\,\frac{L^*(n)}{n^{a+2}}\Big).
\end{aligned}$$

Finally, we have:

$$f_n(a,z)=I_1+I_2\sim I_{12}\sim(z+1)^{n+a+1}\,\frac{L_n^*}{(nz)^{a+1}},\quad n\to\infty,\quad |z+1|>1,\quad a\ge 0.$$

It is not difficult to see that the identity

$$\sum_{k\le n}\frac{L_k^*}{k^a}\binom{n}{k}z^k\equiv nz\cdot f_{n-1}(a,z),$$

implies the statement from the Theorem A.

Next, we compare our results with the well-known test (M. Vuilleumier, [**6**]) for slow variation, i.e.

Proposition M. *Necessary and sufficient conditions for validity of*

$$\sum_{k=1}^\infty a_{nk}L_k\sim AL_n,\quad n\to\infty,$$

for any slowly varying sequence (L_k), *are*

$$(1)\quad \sum_{k>n} |a_{nk}|k^{\eta} = O(n^{\eta}); \qquad (2)\quad \sum_{k\le n} |a_{nk}|k^{-\eta} = O(n^{-\eta}),$$

for some $\eta > 0$;

$$(3)\quad \lim_n \sum_{k\le n} a_{nk} = A.$$

We apply Theorem A to the cited Proposition in the following way.

Let us define a triangular matrix (a_{nk}) by

$$a_{nk} = \begin{cases} \binom{n}{k}\dfrac{n^a}{k^a}\dfrac{z^k}{(z+1)^n} & (|z+1|>1,\ 1\le k\le n), \\ 0 & (k>n). \end{cases}$$

Then,

$$\sum_{k\le n} a_{nk} = \frac{n^a}{(z+1)^n}\sum_{k\le n}\binom{n}{k}\frac{1}{k^a}\,z^k,$$

so, using Theorem A with $L_k^* \equiv 1$, we see that the condition (3) of the Proposition M is satisfied with $A = (1+1/z)^a$, $a \ge 0$.

Validity of the condition (1) is obvious since matrix (a_{nk}) is triangular. Since, according to Theorem A,

$$n^{\eta}\sum_{k\le n}|a_{nk}|k^{-\eta} \sim (1+1/z)^{a+\eta}\left(\frac{|z|+1}{|z+1|}\right)^n,\quad n\to\infty,$$

it is evident that the condition (2) is satisfied only for $z \in R^+$; hence, we could formulate next theorem:

Theorem B. *Asymptotic relation*

$$\sum_{k\le n}\frac{\ell_k}{k^a}\binom{n}{k}z^k \sim \frac{\ell_n}{n^a}(1+1/z)^a(z+1)^n,\quad |z+1|>1,\ a\ge 0,\ n\to\infty,$$

is valid

a) *for every slowly varying sequence* (ℓ_k) *if and only if* $z \in \mathbf{R}^+$;

b) *for any other* $z \in \mathbf{Z}$ *if s.v. sequence* (ℓ_k) *is from the* C^∞ *class* L^*; (*which show its extraordinarity*).

It is worthy to mention here that on or inside the circle $|z+1| = 1$ our sum have entirely different behaviour, i.e. (see [**7**]):

For $|z+1| \le 1,\ z \ne 0,\ a > 0$,

$$\sum_{k\le n}\frac{1}{k^a}\binom{n}{k}z^k = -\frac{\ln^a n}{\Gamma(a+1)} - \frac{\ln^{a-1} n}{\Gamma(a)}\,(\ln(-z)+\gamma+o(1)),\ n\to\infty,$$

where γ is Euler's constant.

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