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ILL-CONDITIONEDNESS AND INTERIOR-POINT METHODS

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In this paper we construct a family of degenerate linear programming problems which cause difficulties to various IPM codes, such as PCx, HOPDM, etc. A theoretical explanation is offered and a possible way out is outlined.

1. INTRODUCTION

As it is well-known, since the discovery of the interior-point methods linear programming (LP) is no longer synonymous with the celebrated simplex method. The interior-point methods (IPMs) have not only a better complexity bound than the simplex method (polynomial vs. exponential) but also enjoy practical efficiency and can be considerably faster than the simplex method for many (but not for all) large scale problems. It is the purpose of this paper to demonstrate that in the case of degeneracy, the interoir-point methods can be seriously affected by ill-conditioning, even in their most robust implementations, such as PCx [**5**], HOPDM [**2**], etc. For the detailed description and discussion of various IPMs see e.g. [**1**], [**2**], [**4**], [**5**].

In order to fix some notation, let us consider the following linear programming problem

$$\min\, c^T x \quad \text{s.t.} \quad Ax = b,\ x \geq 0, \tag{1}$$

with $c, x \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, $A \in \mathbf{R}^{m\times n}$ and its dual

$$\max\, b^T y \quad \text{s.t.} \quad A^T y + s = c,\ s \geq 0, \tag{2}$$

with $s \in \mathbf{R}^n$ being a slack variable and $y \in \mathbf{R}^m$. The feasible sets of (1) and (2) will be denoted by X and Y, respectively. Further, let X^* and Y^* be the

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sets of optimal solutions to problems (1) and (2). A remarkable property of the pair (1), (2) is that if $X^* \neq \emptyset$, $Y^* \neq \emptyset$, then there exists at least one strictly complementary pair of optimal solutions, i.e. an optimal pair x^*, (y^*, s^*), with the property $x^* + s^* > 0$. Let $B = \{i \,|\, x_i^* > 0\}$, $N = \{i \,|\, s_i^* > 0\}$, and let $A_B(A_N)$ be a submatrix of A with columns whose indices are in $B(N)$. Similar notation will be used for the partitions of x, s and c. It can be shown that then sets X^* and Y^* have the following representations:

$$\begin{aligned} X^* &= \{x \in \mathbf{R}^n \,|\, A_B x_B = b,\ x_B \geq 0,\ x_N = 0\}, \\ Y^* &= \{(y,s) \in \mathbf{R}^m \times \mathbf{R}^n \,|\, s_B = c_B - A_B^T y = 0,\ s_N = c_N - A_N^T y \geq 0\}. \end{aligned} \tag{3}$$

We shall see in Section 2 that the structure of the matrix A_B has a strong influence on the numerical performance of interior-point methods. In Section 3 we shall offer a theoretical explanation of the difficulties encountered.

2. CONSTRUCTION OF DEGENERATE TEST EXAMPLES

The family of examples that we consider in this paper will be of the form (1) with nonempty optimal face X^* given by (3). Examples are constructed in such a way that matrix A is of the full rank, A_B is of the type $m \times h$, $h > m$ and $\operatorname{rank} A_B = r$, $r < m$, $c_B = 0$, $c_N > 0$. It is clear that then the optimal objective function value is zero and that the optimal face X^* is of the dimension $h - r > 2$.

Let

$$T = \begin{bmatrix} t_1 & \cdots & t_{k_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & t_{k_1+1} & \cdots & t_{k_2} & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & t_{k_{r-1}+1} & \cdots & t_h \end{bmatrix}_{r \times h}$$

$U = \operatorname{diag}\{u_1, \ldots, u_{n-h-q}\}$, Q is a matrix of the type $(n-h-q) \times q$ and set $m = n - h - q + r$. Furthermore, let

$$A_B = \begin{bmatrix} T \\ 0 \end{bmatrix}_{m \times h}, \quad A_N = \begin{bmatrix} 0 & 0 \\ U & Q \end{bmatrix}_{m \times (n-h)}, \quad \tilde{b} = [\ b_1 \ \cdots \ b_r \ \ 0 \ \cdots \ 0\ ]^T.$$

It is easy to select T, U, Q and $\tilde{b}$ such that the system

$$A_B x_B + A_N x_N = \tilde{b} \tag{4}$$

has a strictly positive solution. Moreover, by selecting e.g. $t_1, \ldots, t_{k_1}$ to be "small" and positive and $\tilde{b}_1$ "large" we assure that any optimal solution has at least one "large" component. This can also be achieved by adding a "large" lower bound on some variables. Similarly, if e.g. $t_{k_1+1}, \ldots, t_{k_2}$ are taken to be "large" and positive and $\widetilde{b}_2$ "small", any optimal solution will have at least one "small" nonzero component. Having large and small nonzero components at the optimum is important in order to avoid termination of IPM codes due to early rounding procedures. Finally, if e.g. the set $\{t_{k_{r-1}}, \dots, t_h\}$ contains both positive and negative numbers and $\widetilde{b}_r = 0$, then the optimal set X^* will be unbounded. Let us note here that any homogeneous equation of (4) can be added to the objective function without changing the optimal set or the optimal objective function value (zero). That was included in our construction as an optional step.

The system (4) has a very special structure that is likely to be exploited by any resonable implementation of IPMs. To avoid that we multiply system (4) from the left by a nonsingular matrix R of order m and permute the variables. More precisely, we set

$$A = R[\,A_B \quad A_N\,]L, \;\; b = R\,\widetilde{b}, \;\; c^T = [\,c_B^T \quad c_N^T\,]L, \tag{5}$$

where L is a permutation matrix of order n.

The described construction is illustrated by the following:

Example 1. Take $n = 20$, $r = 4$, $h = 15$, $q = 2$, which implies $m = 7$. Let

$$c_B = 0, \quad c_N = [9992 \quad 3 \quad 992 \quad 5976 \quad 9989]^T$$

and let

$$T = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -9978 & 9687 & 9993 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9002 & 8994 & 9971 & 3 & 9733 & 3993 \end{bmatrix},$$

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 9981 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & -4891 \\ -1 & 0 \\ -4985 & 1 \end{bmatrix}, \quad \widetilde{b} = [\,8990 \;\; 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \;\; 0\,]^T.$$

Choose

$$R = \begin{bmatrix} -1 & 1 & 1 & -1 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 & -1 \end{bmatrix}$$

and set L to be the permutation matrix with nonzero entries at positions $(1,8)$, $(2,3)$, $(3,7)$, $(4,16)$, $(5,17)$, $(6,18)$, $(7,1)$, $(8,19)$, $(9,20)$, $(10,11)$, $(11,9)$, $(12,13)$, $(13,12)$, $(14,4)$, $(15,5)$, $(16,6)$, $(17,10)$, $(18,15)$, $(19,2)$, $(20,14)$. Let now A, b, c be as defined by (5) and add to $c^T x$ the left-hand side of the second and the third equations of the system $Ax = b$ multiplied by 1 and 9800, respectively. Finally, let us additionally introduce the lower-bound type constraints $x_1 \geq 9728$ and $x_{16} \geq 8829$.

The resulting problem is a small-size problem ($m = 7$, $n = 20$), with integer data in the interval $[-9999, 9999]$. The optimal objective function value is zero, attained e.g. at $(9728, 0, 0, 0, 0, 0, 7990, 1000, 0, 0, 0, 1/3, 0, 0, 0, 9687, 9978, 0, 0, 9728)$. However, this problem could not be solved using the well known IPM codes PCx (public domain version) and HOPDM (Version 2.20). Namely, PCx without iterative refinment stops with UNKNOWN status after 9 iterations offering objective function value - 1688; when iterative refinment is turned on the algorithm exits with the same status but with the objective function value 4405. HOPDM stops after 4 iterations with SUBOPTIMAL status and the objective function value 15375. The same problem was tackled by Microsoft Excel 8.0a Simplex implementation, which reached the optimal value 0 in 12 iterations.

Many other LP problems based on the described principles were constructed; Table 1 summarizes some of the "worst" examples. The first three columns are selfexplanatory; column four (1.b.) indicates whether a positive lower bound was imposed on some of the variables; "yes" ("no") in column five (m.o.f.) tells whether the described modification of the objective function using homogeneous constraints was (was not) performed. For HOPDM, PCx (with and without iterative refinment) and Excel, Table 1 indicates the number of iterations before termination and the best objective function value obtained. All examples have at most four digit integer coefficients.

Table 1: Performance of HOPDM, PCx and Excel on a family of highly degenerate problems

No.	m	n	l.b.	m.o.f.	HOPDM		PCx				EXCEL	
							Without I. R.		With I. R.			
					It.	f	It.	f	It.	f	It.	f
1	7	20	yes	yes	4	1.5e+4	9	–1.7e+3	9	4.4e+3	12	0
2	15	30	no	no	4	3.9e+1	12	6.2e–2	3	4.8e–2	9	0
3	7	20	yes	yes	6	–7.1e+1	17	–2.2e+3	14	3.9e+3	12	–1.5e–8
4	7	20	yes	yes	5	5.6e+4	8	–8.4e+2	12	–8.6e+2	9	0
5	7	20	yes	yes	4	1.6e+5	12	–1.6e+3	13	1.8e+3	12	0
6	7	20	yes	yes	3	1.3e+3	12	2.5e+3	13	7.3e+2	14	0
7	7	20	yes	yes	4	1.1e+4	7	2.5e+3	8	2.5e+3	9	0
8	7	20	yes	yes	5	4.2e+4	6	5.4e+1	11	7.8e+0	9	0

3. THEORETICAL ANALYSIS

In [**3**] it has been shown that the numerical behaviour of interior point methods for LP problems in standard form depends on properties of the matrix A_B which characterizes the optimal face X^* of the (primal) feasible set. The optimal face X^* is called degenerate if rank $A_B < m$. The following theorem was proved in [**3**]:

Theorem 1. *Assume that the optimal face X^* is nonempty and degenerate and suppose that the sequence $\{(x^k, y^k, s^k)\}$ generated by a primal-dual interior-point method has all accumulation points contained in* $\operatorname{ri} X^* \times \operatorname{ri} Y^*$.

Let $D_k = \operatorname{diag}\{(x_1^k/s_1^k)^{1/2}, \ldots, (x_n^k/s_n^k)^{1/2}\},\ k = 1, 2, \ldots$ *Then*

$$\operatorname{cond}(AD_k^2A^T) \to \infty \ \text{ as } \ k \to \infty.$$

Since IPMs for each k solve a system of normal equations with the matrix $AD_k^2A^T$, a large condition number is a strong indication that ill-conditioning may occur. In order to avoid such problems, in [**3**] we propose occasional reformulations of the original problem for which no large condition numbers appear at all. The idea is to obtain a matrix which can be partitioned into well conditioned submatrices. More precisely, if the procedure is applied at the step k the transformed matrix $\widehat{A}$ has the form

$$\widehat{A} = \begin{bmatrix} P & Q \\ 0 & R \end{bmatrix},$$

where P is $\widehat{r} \times \widehat{h}$ matrix, 0 is a zero matrix of size $(m - \widehat{r}) \times \widehat{h}$, $\operatorname{rank} P = \widehat{r}$, Q and R are matrices of size $\widehat{r} \times (n - \widehat{h})$ and $(m - \widehat{r}) \times (n - \widehat{h})$ respectively. The transformation is done by partial pivoting and column permutation and it is not computationaly expensive. The first $\widehat{h}$ columns in $\widehat{A}$ correspond to "large" columns of the scaling matrix D_k. The system of normal equations now has the matrix

$$\widehat{A}\widehat{D}_k^2\widehat{A}^T = \begin{bmatrix} F_k & G_k \\ G_k^T & H_k \end{bmatrix}.$$

If such reformulations of the original problem are performed periodically it has been proved in [**3**] that $\operatorname{cond}(F_k) \le C$ and $\operatorname{cond}(H_k - G_k^T F_k^{-1} G_k) \le C$, where C does not depend on k. This result suggests the following stable implementation of Gaussian elimination for solving the system of normal equations:
The first $\widehat{r}$ pivots should be chosen in the first $\widehat{r}$ columns of the matrix $\widehat{A}\widehat{D}_k^2\widehat{A}^T$ and the elimination procedure should then proceed in the usual way (see [**3**] for more details).

4. CONCLUSIONS

The paper shows that IPMs are sensitive to degeneracy, even in the case of otherwise robust omplementations like PCx, HOPDM, etc. The conclusion is supported by numerical evidence. The reasons for such numerical behaviour have been identified and a simple way out has been uotlined.

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