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ON AN INEQUALITY OF FINK

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The following interesting inequality is due to A. M. Fink [**1**]:

Theorem 1. If $f > 0$ and $\log f$ is convex on $\mathbf{R}$, then

$$(1)\qquad \int_{-1}^{1} f(x+vt)\cos\frac{\pi t}{2}\,\mathrm{d}t \le \frac{2}{\pi}\big(f(x+v)+f(x-v)\big)\qquad (x,v\in\mathbf{R})$$

The aim of this note is to prove that relation holds true if f is convex on $\mathbf{R}$.

First remark the well known fact that log-convexity implies convexity. Indeed, if $g : I \to \mathbf{R}$ ($I \subset \mathbf{R}$, interval) is a strictly positive, log-convex function, then

$$\log g\big(\lambda a+(1-\lambda)b\big) \le \lambda \log g(a) + (1-\lambda)\log g(b)$$

for all $\lambda \in [0,1]$; $a, b \in I$, implying

$$g\big(\lambda a+(1-\lambda)b\big) \le \big(g(a)\big)^{\lambda}\big(g(b)\big)^{1-\lambda}.$$

By Hölder's inequality (see e.g. [**2**]) one has

$$\big(g(a)\big)^{\lambda}\big(g(b)\big)^{1-\lambda} \le \lambda g(a) + (1-\lambda)g(b),$$

since $\lambda + (1-\lambda) = 1$, $\lambda \ge 0$. Thus, g is convex.

To prove (1) for convex f, first note that

$$(2)\qquad I = \int_{-1}^{1} f(x+vt)\cos\frac{\pi t}{2}\,\mathrm{d}t = \int_{0}^{1}\big(f(x+vt)+f(x-vt)\big)\cos\frac{\pi t}{2}\,\mathrm{d}t.$$

Put

$$(3)\qquad g_{x,v}(t) = f(x+vt) + f(x-vt),\ \ t \in [0,1].$$

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Now, since

$$x \pm vt = \left(\frac{1+t}{2}\right)(x \pm v) + \left(\frac{1-t}{2}\right)(x \mp v),$$

by convexity of f one can write

$$\begin{aligned} g_{x,v}(t) \le \frac{1+t}{2} f(x+v) + \frac{1-t}{2} f(x-v) + \frac{1+t}{2} f(x-v) + \frac{1-t}{2} f(x+v) \\ = f(x+v) + f(x-v), \ \ t \in [0,1]. \end{aligned}$$

So by (2) we have

$$I \le \big(f(x+v) + f(x-v)\big) \int_0^1 \cos \frac{\pi t}{2} \, \mathrm{d}t = \frac{2}{\pi} \big(f(x+v) + f(x-v)\big).$$

Equality occurs only when $g_{x,v}$ is linear. Then from (3) it follows that f must be a constant. The given proof shows that the following generalization of (1) is valid.

Theorem 2. *If f is a convex function on $\mathbf{R}$, and c is a nonnegative, even function on $[-1,1]$, then*

$$\int_{-1}^{1} f(x+vt)\, c(t) \, \mathrm{d}t \le \frac{f(x+v) + f(x-v)}{2} \int_{-1}^{1} c(t) \, \mathrm{d}t; \ \ x, v \in \mathbf{R}.$$

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