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ON MAPPINGS WITH CONSERVATIVE DISTANCES AND THE MAZUR–ULAM THEOREM

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In the present paper, two cases concerning the A. D. Aleksandrov problem for a unit distance preserving mapping between Euclidean spaces with dimension one or between strictly convex normed vector spaces are discussed. In addition mappings which preserve distances are also considered.

Let X and Y be normed real vector spaces. Consider the following conditions for a mapping $f : X \to Y$ distance one preserving property (DOPP) and strong distance one preserving property (SDOPP) (cf. [**6**]).

(DOPP) *Let* $x, y \in X$ *with* $\|x - y\| = 1$. *Then* $\|f(x) - f(y)\| = 1$.
(SDOPP) *Let* $x, y \in X$. *Then* $\|f(x) - f(y)\| = 1$ *if and only if* $\|x - y\| = 1$.

Rassias and Semrl [**6**] using the definition of (SDOPP) proved the following

Theorem 1 ([**6**]). *Let* X *and* Y *be real normed vector spaces such that one of them has dimension greater than one. Suppose that* $f : X \to Y$ *is a surjective mapping satisfying* (SDOPP). *Then* f *is an injective mapping satisfying*

$$\big|\|f(x) - f(y)\| - \|x - y\|\big| \le 1 \quad \textit{for all } x, y \in X.$$

Moreover, f *preserves distance* n *in both directions for any positive integer* n.

They pointed out that the condition "one of the spaces has dimension greater than one" can not be relaxed in Theorem 1. If $f : X \to Y$ is a contractive mapping, they obtained the following

Theorem 2 ([**6**]). *Let* X *and* Y *be real normed vector spaces such that one of them has dimension greater than one. Suppose that* $f : X \to Y$ *is a Lipschitz mapping with* $k = 1$, *i.e.*

$$\|f(x) - f(y)\| \le \|x - y\| \quad \textit{for all } x, y \in X.$$

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Assume also that f is a surjective mapping satisfying (SDOPP). *Then f is a linear isometry up to translation.*

Theorem 2 extends the MAZUR–ULAM theorem.

For the special case in Theorem 2 where $X = Y = \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ is a contractive mapping, the following theorem can be proved

Theorem 3. *Suppose that $f : \mathbf{R} \to \mathbf{R}$ is a Lipschitz mapping with $k = 1$, i.e.*

$$|f(x) - f(y)| \le |x - y| \quad \textit{for all } x, y \in \mathbf{R}.$$

Assume that f is a mapping satisfying (DOPP). *Then f is a linear isometry up to translation.*

Proof. Without loss of generality, assume $f(0) = 0$. Otherwise, we substitute $f(x) - f(0)$ for $f(x)$ for all $x \in \mathbf{R}$. Since f satisfies (DOPP), then $|f(1) - f(0)| = |1 - 0| = 1$. Thus $f(1) = 1$ or $f(1) = -1$.

(i) In case $f(1) = 1$, firstly, we will prove by induction that $f(n) = n$ for all nonnegative integers.

Suppose $f(m) = m$ for any integer m with $0 \le m < n$, where n is a positive integer greater than one. Since $|f(n) - f(n-1)| = |f(n) - (n-1)| = 1$, then $f(n) = n - 2$ or $f(n) = n$. Assume $f(n) = n - 2$. Let $r = \dfrac{(n-2) + (n-1)}{2}$. By the contractive property of f, we obtain

$$\begin{aligned}\frac{1}{2} \ge |f(r) - f(n-2)| &\ge |f(n-1) - f(n-2)| - |f(r) - f(n-1)| \\ &\ge 1 - |r - (n-1)| = \frac{1}{2}\,.\end{aligned}$$

Therefore $|f(r) - f(n-2)| = 1/2$. Applying the same argument, it follows that $|f(r) - f(n-1)| = 1/2$. Since $f(n-2) = n-2$ and $f(n-1) = n-1$, then

$$f(r) = \frac{(n-2) + (n-1)}{2}\,.$$

Set $s = \dfrac{(n-1) + n}{2}$. Similarly, we obtain

$$|f(s) - f(n-1)| = \frac{1}{2}, \quad |f(s) - f(n)| = \frac{1}{2} \quad \text{and} \quad f(s) = \frac{(n-2) + (n-1)}{2}.$$

Thus $|f(r) - f(s)| = 0$ and $|r - s| = 1$, which contradicts the fact that f satisfies (DOPP). Hence $f(n) = n$ for any nonnegative integer n.

Similarly, $f(-1)$ must take the value -1. Otherwise $f(-1) = 1$ and $f(-1/2) = f(1/2) = 1/2$, which contradicts the fact that f satisfies (DOPP). By induction, the proof follows for $f(n) = n$ for any negative integer n exactly for the case of nonnegative integers we proved before.

For any $x \in \mathbf{R}$, there exists an integer n_0 such that $x \in [n_0, n_0+1]$. Since $f : \mathbf{R} \to \mathbf{R}$ is a LIPSCHITZ mapping with $k = 1$, by the same argument as above, it is true that

$$|f(x) - f(n_0)| = |x - n_0| \qquad \text{and} \qquad |f(x) - f(n_0+1)| = |x - n_0 - 1|.$$

Therefore together with the fact that $f(n) = n$ for any integer n, f must satisfy the property $f(x) = x$ for all $x \in \mathbf{R}$, and thus f is a linear isometry.

(ii) We will examine the case $f(1) = -1$. Let $g(x) = -f(x)$ for all $x \in \mathbf{R}$. Then g satisfies (DOPP) and it is also a Lipschitz mapping with $k = 1$. By (i), g must be a linear isometry and $g(x) = x$ for all $x \in \mathbf{R}$. Therefore $f(x)$ is a linear isometry and $f(x) = -x$ for all $x \in \mathbf{R}$. $\square$

Since any real normed vector space with dimension one is linearly isometric to $\mathbf{R}$, then according to Theorem 2 and Theorem 3, it follows

Corollary 4. *Let X and Y be real normed vector spaces. Suppose that $f : X \to Y$ is a Lipschitz mapping with $k = 1$:*

$$\|f(x) - f(y)\| \leq \|x - y\| \quad \textit{for all } x, y \in X.$$

Assume also that f is a surjective mapping satisfying (SDOPP). *Then f is a linear isometry up to translation.*

If $f : X \to Y$ preserves two distances instead of one distance, BENZ and BERENS in [**3**] proved the following:

Theorem 5 ([**3**]). *Let X and Y be real normed vector spaces. Assume that* $\dim X \geq 2$ *and Y is strictly convex. Suppose $f : X \to Y$ satisfies the property that for all $x, y \in X$ with $\|x - y\| = \rho$, then $\|f(x) - f(y)\| \leq \rho$; and for all $x, y \in X$ with $\|x - y\| = m\rho$, then $\|f(x) - f(y)\| \geq m\rho$, where m is a positive integer greater than one. Then f is a linear isometry up to translation.*

If f preserves two distances with a noninteger ratio, and X and Y are real normed vector spaces such that Y is strictly convex and $\dim Y \geq 2$, it is an *open problem* whether or not f must be an isometry (see [**5**]). However, if f preserves three distances, one can prove the following

Theorem 6. *Let X and Y be real normed vector spaces. Assume that* $\dim X \geq 2$ *and Y is strictly convex. Suppose $f : X \to Y$ satisfies the property that f preserves the three distances 1, a and $1 + a$, where a is any positive constant. Then f is a linear isometry up to translation.*

Proof. (i) Let $x, y \in X$ with $\|x - y\| = 2 + a$. Set

$$x_1 = x + \frac{1}{2+a}(y - x), \quad x_2 = x + \frac{1+a}{2+a}(y - x).$$

Then

$$\|x_1 - x\| = 1,\ \|x_1 - x_2\| = a,\ \|y - x_1\| = 1 + a,\ \|x_2 - x\| = 1 + a,\ \|y - x_2\| = 1.$$

It follows that

$$\|f(x_1)-f(x)\|=1,\quad \|f(x_1)-f(x_2)\|=a,\quad \|f(y)-f(x_1)\|=1+a,$$
$$\|f(x_2)-f(x)\|=1+a,\quad \|f(y)-f(x_2)\|=1,$$

and

$$\|f(x_2)-f(x)\|=\|f(x_1)-f(x)\|+\|f(x_2)-f(x_1)\|=1+a,$$
$$\|f(y)-f(x_1)\|=\|f(x_2)-f(x_1)\|+\|f(y)-f(x_2)\|=1+a.$$

Since Y is strictly convex, then

$$f(x_1)=f(x)+\frac{1}{1+a}\big(f(x_2)-f(x)\big),\quad f(x_2)=f(x_1)+\frac{a}{1+a}\big(f(y)-f(x_1)\big).$$

Hence

$$f(x)=\frac{1+a}{a}f(x_1)-\frac{1}{a}f(x_2)\qquad\text{and}\qquad f(y)=\frac{1+a}{a}f(x_2)-\frac{1}{a}f(x_1).$$

Thus $\|f(y)-f(x)\|=2+a$ for all $x,y\in X$ with $\|x-y\|=2+a$. Therefore f also preserves the distance $2+a$.

(ii) Let $x,y\in X$ with $\|x-y\|=2+2a$. Set

$$x_1=x+\frac{1+a}{2+2a}(y-x),\quad x_2=x+\frac{2+a}{2+2a}(y-x).$$

Then

$$\|x_1-x\|=1+a,\ \|x_1-x_2\|=1,\ \|y-x_1\|=1+a,\ \|x_2-x\|=2+a,\ \|y-x_2\|=a.$$

Since f preserves distances 1, a, $1+a$ and $2+a$, in a similar way, we obtain that

$$\|f(y)-f(x)\|=2+2a.$$

Hence the mapping f preserves the distance $2+2a$.

By (i), (ii) and Theorem 5, f must be an isometry up to translation.

Remark 1. In Theorem 6, the result is obviously true if the distances 1, a and $1+a$ are substituted by a, b and $a+b$, where a and b are positive constants. In particular, if the ratio of the two positive constants a and b is an integer, then f is a linear isometry up to tranlation due to a theorem of Benz and Berens [**3**].

Theorem 7. *Let X and Y be real normed vector spaces. Assume that Y is strictly convex. Suppose $f:X\to Y$ satisfies* (DOPP) *and is a Lipschitz mapping with $k=1$, i.e.*

$$\|f(x)-f(y)\|\le\|x-y\|\quad \textit{for all } x,y\in X.$$

Then f is a linear isometry up to translation.

Proof. Let $x, y \in X$ with $\|x-y\| = 1/2$. Set $z = x + 2(y-x)$. Then

$$\|x-z\| = 1, \qquad \|z-y\| = \frac{1}{2}\,,$$

and

$$\begin{aligned}\|x-y\| &\geq \|f(x)-f(y)\| \geq \|f(z)-f(x)\| - \|f(z)-f(y)\| \\ &\geq 1 - \|z-y\| \geq \frac{1}{2}\,.\end{aligned}$$

Hence $\|f(x)-f(y)\| = 1/2$. Similarly $\|f(z)-f(y)\| = 1/2$ and

$$\|f(z)-f(x)\| = \|f(z)-f(y)\| + \|f(y)-f(x)\| = 1.$$

Because of the fact Y is strictly convex, then

$$f(y) = \frac{f(x)+f(z)}{2} \qquad \text{and} \qquad \|f(y)-f(x)\| = \frac{1}{2}.$$

Hence f preserves distances 1 and $1/2$. Applying a similar method as in the proof of Theorem 6, the mapping f preserves distance n in both directions for any positive integer n. Since f is a contractive mapping, it is easy to verify that f is an isometry due to the proof of Theorem 5 (see [**6**]). Therefore f is a linear isometry up to translation due to a theorem of Baker [**1**]. □

A generalization of Theorem 7 was obtained independently and before us by Yumei [**7**] in the following form:

Theorem 8. *Let X and Y be real normed vector spaces. Assume that Y is strictly convex. Suppose $f : X \to Y$ satisfies* (DOPP) *and*

$$\|f(x)-f(y)\| \leq \|x-y\| \quad \textit{whenever } x, y \in X \textit{ satisfy } \|x-y\| \leq 1.$$

Then f is a linear isometry up to translation.

Remark 2. G. Ding has pointed out to us that in Theorem 8, the condition

$$\text{“}\ \|f(x)-f(y)\| \leq \|x-y\| \quad \text{whenever } x, y \in X \text{ satisfy } \|x-y\| \leq 1.\text{ ”}$$

can be replaced by the condition “*for some positive constant β, the mapping f satisfies the property:*

$$\|f(x)-f(y)\| \leq \|x-y\| \quad \textit{whenever } x, y \in X \textit{ satisfy } \|x-y\| \leq \beta\text{ ”}$$

and that still f is also a linear isometry up to translation. Beacause of the fact, we can choose a positive integer n such that $n\beta > 1$, it can be shown that f preserves distance $1/n$.

If $f:\mathbf{R}^n\to\mathbf{R}^m$ preserves some distance, it follows that $n\le m$. This is true because $\mathbf{R}^m$ has equilateral *n–simplices* if and only if $n\le m$. RASSIAS [**4**] has proved the following result:

Theorem 9 ([**4**]). *For any integer $n\ge 1$, there exists an integer n_m such that $N\ge n_m$ implies that there exists a map $f:\mathbf{R}^n\to\mathbf{R}^N$ which satisfies the condition* (DOPP) *but is not an isometry.*

A counterexample that $f:\mathbf{R}^2\to\mathbf{R}^8$ satisfies (DOPP) but f is not an isometry has been given in [**4**]. That is, for arbitrary three points p_1, p_2, p_3 in $\mathbf{R}^2$ forming an equilateral triangle with unit length, then $f(p_1)$, $f(p_2)$ and $f(p_3)$ also form an equilateral triangle with unit length, but f is not an isometry.

However, for arbitrary three points p_1, p_2, p_3 in $\mathbf{R}^2$ forming an isosceles triangle with the ratio between the base with unit length and the height of the triangle or the ratio between the oblique line with unit length and the height of the triangle being an even integer, if $f(p_1)$, $f(p_2)$ and $f(p_3)$ also make an isosceles triangle with the same size, then f must be a linear isometry up to translation due to the following theorem.

Theorem 10. *Let X and Y be real Hilbert spaces with* $\dim X\ge 2$. *Suppose $f:X\to Y$ satisfies the property that*

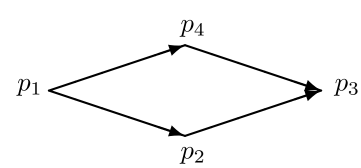
(i) *f preserves the distances a, b with the ratio of a and b being $\dfrac{\sqrt{4n^2-1}}{n}$ for some positive integer n; or*

(ii) *f preserves the distances a, b with the ratio of a and b being $\dfrac{\sqrt{n^2+1}}{2n}$ for some positive integer n with $n>1$.*

Then f is a linear isometry up to translation.

Proof. (i) Without loss of generality, suppose that f preserves the two distances $\sqrt{4n^2-1}$ and n. For the case $n=1$, the property follows due to a theorem of XIANG [**8**].

For the case $n>1$, let p_2, p_4 in X with $\|p_4-p_2\|=1$. Since $\dim X\ge 2$, then one can select p_1, p_3 in X such that p_1,p_2,p_3,p_4 in X form a parallelogram with $\|p_2-p_1\|=\|p_3-p_4\|=\|p_3-p_2\|=\|p_4-p_1\|=n$, $\|p_1-p_3\|=\sqrt{4n^2-1}$, $\|p_2-p_4\|=1$



(1 − 1)

Because of the fact that f preserves the distances $\sqrt{4n^2-1}$ and n, then

$$\begin{aligned}\|f(p_2)-f(p_1)\|&=\|f(p_3)-f(p_2)\|\\&=\|f(p_4)-f(p_3)\|=\|f(p_4)-f(p_1)\|=n,\end{aligned}$$

and $\|f(p_3)-f(p_1)\|=\sqrt{4n^2-1}$. Let $x=f(p_1)-f(p_2)$, $y=f(p_3)-f(p_2)$ and $z=f(p_4)-f(p_2)$. Then

$$\|x\|=\|y\|=\|z-x\|=\|z-y\|=n,\ \|x-y\|=\sqrt{4n^2-1}\quad\text{and}\quad(x+y,x+y)=1.$$

Hence $(z,x) = (z,y) = \dfrac{1}{2}(z,z)$. If we use the Cauchy–Schwarz inequality, we obtain that

$$\|z\|^2 = (z, x+y) \leq \|z\|\,\|x+y\| = \|z\|.$$

Thus $\|f(p_4) - f(p_2)\| \leq 1$. By Theorem 5, f is a linear isometry up to translation.

(ii) In a similar way, if f preserves the distances a, b with the ratio of a and b being $\sqrt{n^2+1}/(2n)$ for some positive integer n with $n > 1$, it follows that f must be a linear isometry up to translation. □

From Beckman and Quarles [**2**] it follows that if $f : \mathbf{R}^n \to \mathbf{R}^n$ satisfies (DOPP) for some positive integer n with $n \geq 2$, then the distance $\sqrt{2(n+1)/n}$ is also preserved by f. Generally, if $f : X \to Y$ satisfies (DOPP), where X and Y are real Hilbert spaces with $\dim X \geq n$ and $n \geq 2$, let p, q in X with $\|p-q\| = \sqrt{2(n+1)/n}$, then $\|f(p) - f(q)\| \leq \sqrt{2(n+1)/n}$. In fact, suppose p, $p_1, \cdots, p_n$ and q, $p_1, \cdots, p_n$ form two equilateral $n-$ *simplices* with unit side and $\|p-q\| = \sqrt{2(n+1)/n}$. Let $z = f(p) - f(q)$, $x_i = f(p) - f(p_i)$ and $y_i = f(p_i) - f(q)$ for $i = 1, 2, \cdots, n$, then $\|x_i\| = \|y_i\| = \|x_i - x_j\| = 1 (i \neq j)$ and $z = x_i + y_i$. Thus by $(z - x_i, z - x_i) = (z - y_i, z - y_i) = 1$, we can get

$$(z,z) = 2\,(z, x_i) = \frac{2}{n}\left(z, \sum_{i=1}^{n} x_i\right) \leq \frac{2}{n}\left\|\sum_{i=1}^{n} x_i\right\| \cdot \|z\| \leq \sqrt{2(n+1)/n}\,\|z\|.$$

Therefore $\|f(p) - f(q)\| \leq \sqrt{2(n+1)/n}$. Hence by Theorem 5, we obtain the following

Theorem 11. *Let X and Y be real Hilbert spaces with $\dim X \geq n$ and $n \geq 2$. Suppose $f : X \to Y$ satisfies* (DOPP) *and preserves the distance $k\sqrt{2(n+1)/n}$ for some positive integer k with $k \geq 2$. Then f is a linear isometry up to translation.*

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REFERENCES

1. J. A. Baker: *Isometries in normed spaces.* Amer. Math. Monthly **78** (1971), 655–658.
2. F. S. Beckman, D. A. Quarles: *On isometries of Euclidean spaces.* Proc. Amer. Math. Soc. **4** (1953), 810–815.
3. W. Benz, B. Berens: *A contribution to a theorem of Ulam and Mazur.* Aeq. Math. **34** (1987), 61–63.
4. Th. M. Rassias: *Some remarks on isometric mappings.* Facta Univ., Ser. Math. Inform. **2** (1987), 49–52.

5. Th. M. Rassias: *Mappings that preserve unit distance.* Indian J. Math. **32** (1990), 275–278.
6. Th. M. Rassias, P. Semrl: *On the Mazur–Ulam theorem and the Aleksandrov problem for unit distance preserving mappings.* Proc. Amer. Math. Soc. **118** (1993), 919–925.
7. Ma Yumei: *Aleksandrov problem for unit distance preserving mapping.* Acta Math. Sci. (to appear).
8. Shuhuang Xiang: *Aleksandrov problem and mappings which preserve distances* in: Functional Equations and Inequalities (ed. Th. M. Rassias), Kluwer Academic Publishers, Dordrecht (to appear).

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