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A NOTE ON CONNECTION BETWEEN P-CONVEX AND SUBADDITIVE FUNCTIONS

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The purpose of this paper is to establish a connection between p-convex and locally subadditive functions.

Primary tools in theory of analytic inequalities are classes of convex and subadditive functions [**4**].

A function $f : \mathbf{R}^n \to \mathbf{R}$ is convex if

$$f(sx + ty) \leq sf(x) + tf(y) \tag{1}$$

for all $x, y \in \mathbf{R}^n$ and all $s, t \in [0, 1]$ with $s + t = 1$.

A function $f : A \to \mathbf{R}(A \subset \mathbf{R}^n, A + A \subset A)$ is called locally subadditive (resp. superadditive) if for all $x, y \in A$:

$$f(x + y) \leq f(x) + f(y) \text{ (resp. } f(x + y) \geq f(x) + f(y) \tag{2}$$

The purpose of our work [**5**] was to establish a connection between those classes of functions. There we proved that every convex $g(x)$ defined on $]a, b[$ $(-\infty \leq a < b \leq +\infty)$ produces a locally subadditive function $f(x, y)$ on $C \subset \mathbf{R}^2$,

$$C = \left\{(x, y) : a < \frac{y}{x} < b, x > 0\right\}$$

given with:

$$f(x, y) = x \cdot g\left(\frac{y}{x}\right). \tag{3}$$

A generalization of this proposition for function on $\mathbf{R}^m$, is given in following proposition 3. In this artivcle we treat so called p-convex function as a source of an enlarged class of subadditive functions in given explicit form in $\mathbf{R}^2$, which is also capable of great generalizations.

A function $f : A \to \mathbf{R}$ (A is a cone in $\mathbf{R}^n$) is p-convex for some $p \in]0, 1[$ if

$$f(sx + ty) \leq s^p f(x) + t^p f(y), \tag{4}$$

for all $x, y \in A$ and all $s, t \in]0, 1[$ with $s + t = 1$.

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This definition shows wider notion of convexity, evidently, every positive convex function is p-convex, but the converse is not true. For example, function $f(x) = x^p$, $0 < p < 1$, $x > 0$, is not convex but is p-convex:

$$f(sx+ty) = (sx+ty)^p \leq (sx)^p + (ty)^p = s^p f(x) + t^p f(y). \tag{5}$$

Also, every positive p_2-convex is p_1-convex function for $0 < p_1 < p_2 \leq 1$.

A function $f : A \to \mathbf{R}$ (A is a cone in $\mathbf{R}^n$) is positive homogenous with degree p, if $f(tx) = t^p f(x); t, p \in \mathbf{R}^+$.

In propositions 1 and 2 we are dealing with necessary and suficient conditions for $f : C \to \mathbf{R}$ to be subadditive, depending of given $g :]a,b[\to \mathbf{R}$. In proposition 3 we give possible generalization in the case $p = 1$.

Proposition 1. *Let $g :]a,b[\to \mathbf{R}$ be p-convex function. Then*

$$f(x,y) = x^p \cdot g\left(\frac{y}{x}\right) \tag{6}$$

is positive homogenous of degree p subadditive function on

$$C = \left\{(x,y) : a < \frac{y}{x} < b, x > 0\right\}$$

Proof. Let $(x_i, y_i) \in C$, $i = 1, 2$; then

$$\begin{aligned}
f((x_1,y_1)+(x_2,y_2)) &= f(x_1+x_2, y_1+y_2) = (x_1+x_2)^p \cdot g\left(\frac{y_1+y_2}{x_1+x_2}\right) \\
&= (x_1+x_2)^p \cdot g\left(\frac{x_1}{x_1+x_2}\cdot\frac{y_1}{x_1} + \frac{x_2}{x_1+x_2}\cdot\frac{y_2}{x_2}\right) \\
&\leq (x_1+x_2)^p \cdot \left(\frac{x_1}{x_1+x_2}\right)^p \cdot g\left(\frac{y_1}{x_1}\right) \\
&\quad + (x_1+x_2)^p \cdot \left(\frac{x_2}{x_1+x_2}\right)^p \cdot g\left(\frac{y_2}{x_2}\right) \\
&= f(x_1,y_1) + f(x_2,y_2),
\end{aligned}$$

i.e. $f(\cdot)$ is subadditive on C. That $f(\cdot)$ is positive homogenous of degree p is obvious. □

We conclude that every p-convex function on $\mathbf{R}^+$ produces subadditive function on C. Conversely:

Proposition 2. *Let $f : C \to \mathbf{R}$, $C = \left\{(x,y) : a < \frac{y}{x} < b,\ x > 0\right\}$, be subadditive and positive homogenous function with exact degree p. Then $f(\cdot)$ has to be in the form:*

$$f(x,y) = x^p \cdot g\left(\frac{y}{x}\right), \tag{7}$$

where $g(\cdot)$ is p-convex.

Proof. First, we show that $g(y) := f(1, y)$ is p-convex. Using subadditivity of $f(\cdot)$ we get

$$\begin{aligned} g(sy_1 + ty_2) &= f(1, sy_1 + ty_2) = f(s + t, sy_1 + ty_2) \\ &\leq f(s, sy_1) + f(t, ty_2) = s^p f(1, y_1) + t^p f(1, y_2) \end{aligned}$$

i.e. $g(\cdot) = f(1, \cdot)$ is p-convex. Now, using homogenously (with $t = \frac{1}{x}$) of $f(\cdot)$ we have:

$$\frac{1}{x^p} f(x, y) = f\left(\frac{1}{x} \cdot x, \frac{1}{x} \cdot y\right) = f\left(1, \frac{y}{x}\right) = g\left(\frac{y}{x}\right),$$

i.e.

$$f(x, y) = x^p \cdot g\left(\frac{y}{x}\right)$$

and the proof is over. We are concluding with a generalization (in the case $p = 1$) of proposition cited 2. □

Proposition 3. *A convex function $g : \mathbf{R}^m \to \mathbf{R}$ produces positive homogenous subadditive $f(\cdot)$ on $C \subset \mathbf{R}^2$ given with:*

$$f(x) = \langle A, x\rangle \cdot \left(\frac{\langle B_1, x\rangle}{\langle A, x\rangle}, \frac{\langle B_2, x\rangle}{\langle A, x\rangle}, \ldots, \frac{\langle B_m, x\rangle}{\langle A, x\rangle}\right), \tag{8}$$

where C is half-plane in $\mathbf{R}^m$, i.e. $C = \{x = (x_1, x_2, \ldots, x_m),\ \langle A, x\rangle > 0\}$, $B_i = (B_{i1}, B_{i2}, \ldots, B_{im})$, $i = 1, 2, \ldots, m$; are vectors not equal to zero, $A = (A_1, A_2, \ldots, A_n)$ is constant vector in $\mathbf{R}^n$, and $\langle a, b\rangle$, as usual, defines inner product of $a, b \in \mathbf{R}^n$.

Proof. Since

$$\frac{\langle B_k, x + y\rangle}{\langle A, x + y\rangle} = \frac{\langle A, x\rangle}{\langle A, x + y\rangle} \cdot \frac{\langle B_k, x\rangle}{\langle A, x\rangle} + \frac{\langle A, y\rangle}{\langle A, x + y\rangle} \cdot \frac{\langle B_k, y\rangle}{\langle A, y\rangle},\ k = 1, 2, \ldots, m;$$

using convexity of $g(\cdot)$, we get:

$$\begin{aligned} f(x + y) &= \langle A, x + y\rangle \cdot g\left(\frac{\langle B_1, x + y\rangle}{\langle A, x + y\rangle}, \frac{\langle B_2, x + y\rangle}{\langle A, x + y\rangle}, \ldots, \frac{\langle B_m, x + y\rangle}{\langle A, x + y\rangle}\right) \\ &\leq \langle A, x + y\rangle \cdot s \cdot g\left(\frac{\langle B_1, x\rangle}{\langle A, x\rangle}, \frac{\langle B_2, x\rangle}{\langle A, x\rangle}, \ldots, \frac{\langle B_m, x\rangle}{\langle A, x\rangle}\right) \\ &\quad + \langle A, x + y\rangle \cdot t \cdot g\left(\frac{\langle B_1, y\rangle}{\langle A, y\rangle}, \frac{\langle B_2, y\rangle}{\langle A, y\rangle}, \ldots, \frac{\langle B_m, y\rangle}{\langle A, y\rangle}\right) \\ &= f(x) + f(y); s = \frac{\langle A, x\rangle}{\langle A, x + y\rangle}, t = \frac{\langle A, y\rangle}{\langle A, x + y\rangle}. \end{aligned}$$

The fact that $f(\cdot)$ is positive homogenous ($p = 1$) is evident. □

Proposition 4. *Let* $f : C \to \mathbf{R}$, $C = \{(x_1, x_2, \ldots, x_n) : x_n > 0\}$ *be subadditive and positive homogenous* $(p = 1)$. *Then* $f(\cdot)$ *has to be in the form:*

$$f(x_1, x_2, \ldots, x_n) = x_n \cdot g\left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right) \tag{9}$$

where $g(\cdot)$ *is convex.*

Proof. Similarly as in proposition 2,

$$g(x_1, x_2, \ldots x_{n-1}) := f(x_1, x_2, \ldots, x_{n-1}, 1)$$

is convex. Now, for $t = \dfrac{1}{x_n}$ we obtain:

$$\frac{1}{x_n} \cdot f(x_1, x_2, \ldots x_n) = f\left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, 1\right) = g\left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right),$$

i.e.

$$f(x_1, x_2, \ldots, x_n) = x_n \cdot g\left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right). \quad \square$$

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