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A CONSTRUCTION OF COMPLETELY PRIME SUBSETS ASSOCIATED WITH IDEMPOTENTS OF A SEMIGROUP WITH APARTNESS

Daniel Abraham Romano

This investigation is in constructive algebra. We shall give a construction of completely prime strongly extensional subset $M(e)$ associated with an idempotent e of a semigroup S equiped with an apartness, such that $G_e \subseteq \overline{M(e)}$ and $e\#M(e)$, and we will give some descriptions of the family $\{M(e) : e \in E(S)\}$.

Let $(S, =, \neq, \cdot, 1)$ be a semigroup with apartness ([**6**], [**7**]) where the semigroup operation is strongly extensional in the sense

$$(\forall a, b, x, y \in S)(ax \neq by \Rightarrow a \neq b \wedge x \neq y).$$

Let T be a subset of S. We say that T is a left cosubsemigroup of S (or a right consistent subset of S ([**2**])) if and only if $(\forall x, y \in S)(xy \in T \Rightarrow y \in T)$, the set T is a right cosubsemigroup of S (or a left consistent subset of S ([**2**])) if and only if $(\forall x, y \in S)(xy \in T \Rightarrow x \in T)$, the set T is a cosubsemigroup of S (or a completely prime subset of S ([**2**])) if and only if $(\forall x, y \in S)(xy \in T \Rightarrow x \in T \vee y \in T)$, and the set T is a coideal of S (or a consistent subset of S ([**2**]) iff $(\forall x, y \in S)(xy \in T \Rightarrow x \in T \wedge y \in T)$. The subset T of S is strongly extensional ([**6**], [**7**]) iff $(\forall x, y \in S)(x \in T \Rightarrow x \neq y \vee y \in T)$. Let $a \in S$. By $a\#T$ we denote $(\forall t \in T)(t \neq a)$ and by $\overline{T}$ we denote the set $\{a \in S : a\#T\}$. The subset T of S is detachable iff $(\forall x \in S)(x \in T \vee x\#T)$ ([**1**], [**6**], [**7**]). A relation q on S is a coequality relation on S iff q is consistent, symmetric and cotransitive relation on S ([**3**], [**4**], [**5**]). A coequality relation q on S is a cocongruence ([**3**], [**5**]) or q is compatible with the semigroup operation on S iff $(\forall a, b, x, y \in S)((ax, by) \in q \Rightarrow (a, b) \in q \vee (x, y) \in q)$.

For undefined notions and notations we refer to the books [**1**], [**2**], [**6**] and to the papers [**4**], [**5**].

Semigroups with apartness were first defined and were studied by A. Heyting. W. Ruitenburg studied semigroups with apartness in his dissertation (1982) [**6**]. After that the author of this paper has worked on this important topic in his

1991 Mathematics Subject Classification: 03F55, 20M99 dissertation (1985) [**3**]. Semigroups with apartnesses were studied by A. S. Troelstra and D. van Dalen in their monograph (1988) [**7**]. In this paper we give a construction of cosubsemigroups (completely prime subsets) associated with idempotents of semigroup S and describe some properties of the family of so constructed cosubsemigroups.

We start with following proposition.

Theorem 1. *Let e be an idempotent of a semigroup S with apartness. Then:*

(1) $A(e) = \{a \in S : ae \neq a\}$ *is a strongly extensional right consistent subset of S such that $e\#A(e)$.*

(2) $B(e) = \{b \in S : eb \neq b\}$ *is a strongly extensional left consistent subset of S such that $e\#B(e)$.*

(3) $X(e) = \{a \in S : e\#Sa\}$ *is a strongly extensional left ideal of S such that $e\#X(e)$.*

(4) $Y(e) = \{b \in S : e\#bS\}$ *is a strongly extensional right ideal of S such that $e\#Y(e)$.*

(5) $Z(e) = \{x \in S : e\#SxS\}$ *is a strongly extensional ideal of S such that $e\#Z(e)$.*

Proof.
(1) Let x and a be arbitrary elements of S such that $xa \in A(e)$. Then $xae \neq xa$, whence it follows $a \in A(e)$. So, the set $A(e)$ is a right consistent subset of S. If $a \in A(e)$, then for every $y \in S$ holds $ae \neq a \Rightarrow ae \neq ye \vee ye \neq y \vee y \neq a$. Hence $y \in A(e) \vee y \neq a$ and $A(e)$ is a strongly extensional right consistent subset of S. If we get $a \in A(e)$, we will have $ae \neq a$. Thus $ae \neq e^2 \vee e \neq a$ and $a \neq e$.
(3) Assume $a \in X(e)$. Then $e\#Sa$. As we have $Sxa \subseteq Sa$ for every x in S, we have $e\#Sxa$. Thus $xa \in X(e)$ and the set $X(e)$ is a left ideal of S. Let y be an arbitrary element of S and let a in $X(e)$. Then $e\#Sa$. Thus $e \neq sa (s \in S)$. From this it follows $(\forall t \in T)(e \neq ty \vee ty \neq sa)$. So, $(\forall s \in T)(e \neq sy) \vee y \neq a$. Therefore, the set $X(e)$ is a strongly extensional subset of S. Further, if $a \in X(e)$, then $e\#Sa$. Thus $e \neq a$. □

Corollary 1.1. $A(1) = \emptyset, B(1) = \emptyset$.

Corollary 1.2. $Z(e) \subseteq X(e) \cap Y(e)$.

The next theorem is one of main results of this paper: we will give a construction of a strongly extensional completely prime subset $M(e)$ of S (cosubsemigroup of S) associated with an idempotent $e \in E(S)$, such that $e\#M(e)$.

Theorem 2. *Let e be an idempotent of a semigroup S with apartness. Then the set $M(e) = A(e) \cup B(e) \cup X(e) \cup Y(e)$ is a strongly extensional cosubsemigroup of S such that $e\#M(e)$.*

Proof. (1) Let $ab \in M(e)$. Then $abe \neq ab \vee eab \neq ab \vee e\#Sab \vee e\#abS$. If ab in $A(e)$, then b is in $A(e) \subseteq M(e)$, because $A(e)$ is a right consistent subset of S. If ab in $B(e)$, then a is in $B(e) \subseteq M(e)$, because $B(e)$ is a left consistent subset of S. Assume that $ab \in X(e)$, i.e.$(\forall u \in S)(uab \neq e)$. Then we have the sequence

$$\begin{array}{l} (\forall x, y \in S)(xyab \neq e) \Rightarrow \\ (\forall x, y \in S)(xyab \neq xeb \vee xeb \neq xb \vee xb \neq e) \Rightarrow \\ (\forall x, y \in S)(ya \neq e \vee eb \neq b \vee xb \neq e) \Rightarrow \\ (\forall x \in S)(xb \neq e) \vee eb \neq b \vee (\forall y \in S)(ya \neq e) \Rightarrow \\ a \in Y(e) \subseteq M(e) \vee b \in X(e) \subseteq M(e) \vee b \in B(e) \subseteq M(e). \end{array}$$

Similarly, we have the implication $ab \in Y(e) \Rightarrow a \in M(e) \vee b \in M(e)$. So, $M(e)$ is a cosubsemigroup of S such that $e\#M(e)$.
(2) Let a be an arbitrary element of $M(e) = A(e) \cup B(e) \cup X(e) \cup Y(e)$ and let b be an arbitrary element of S. Then $a \in A(e)$ or $a \in B(e)$ or $a \in X(e)$ or $a \in Y(e)$. Then $a \neq b \vee b \in M(e)$, because sets $A(e)$, $B(e)$, $X(e)$ and $Y(e)$ are strongly extensional in S. □

In the next theorem and few following corollaries we will describe the family $\{M(e) \,:\, e \in E(S)\}$.

Theorem 3. *Let S be a semigroup with apartness and with at least two idempotents. Then*

$$(\forall e, f \in E(S))(e \neq f \Rightarrow M(e) \cup M(f) = S).$$

Proof. Let a be an arbitrary element of semigroup S. Then

$$\begin{array}{l} e \neq f \Rightarrow \\ (\forall x, y \in S)(e \neq ax \vee ax \neq fax \vee fax \neq fe \vee fe \neq yae \vee yae \neq ya \vee ya \neq f) \Rightarrow \\ (\forall x, y \in S)(e \neq ax \vee a \neq fa \vee ax \neq e \vee f \neq ya \vee ae \neq a \vee ya \neq f) \Rightarrow \\ (\forall x \in S)(e \neq ax) \vee a \neq fa \vee (\forall y \in S)(f \neq ya) \vee ae \neq a \Rightarrow \\ a \in Y(e) \vee a \in B(f) \vee a \in X(f) \vee a \in A(e) \Rightarrow a \in M(e) \vee a \in M(f). \end{array}$$ □

Corollary 3.1. *Let e be an idempotent of a semigroup S with apartness. Then $G_e \subseteq \overline{M(e)}$ and $\overline{M(e)}$ is a subsemigroup of S.*

Proof.
(i) We have that $e \in \overline{M(e)}$ because $e\#M(e)$. Let a and b be elements of $\overline{M(e)}$ and let u be an arbitrary element of $M(e)$. Then $u \neq ab$ or $ab \in M(e)$ by strongly extensionality of $M(e)$ in S. As $M(e)$ is a cosubsemigroup of S we have $a \in M(e)$ or $b \in M(e)$. It is impossible. Hence $u \neq ab$ for each $u \in M(e)$. So, $ab\#M(e)$. Therefore, the set $\overline{M(e)}$ is a subsemigroup of S such that $e \in \overline{M(e)}$.
(ii) Let x be an arbitrary element of G_e. Then for an arbitrary element u of $M(e)$ we have $x \neq u$ or $x \in M(e)$. The second case is impossible. So, $x\#M(e)$. Thus $G_e \subseteq \overline{M(e)}$. □

Corollary 3.2. *Let e be an idempotent of a semigroup S with apartness. Then the relation $t(e)$ on S, defined by $(a,b) \in t(e) \Leftrightarrow a \neq b \wedge (a \in M(e) \vee b \in M(e))$, is a coequality relation on S such that*

$$(*)\ a \in M(e) \Rightarrow at(e) = \{x \in S \ :\ x \neq a\},\quad a \in G_e \Rightarrow at(e) = M(e)\ (**).$$

Proof.
(1) Let (u,w) be an arbitrary element of $t(e)$ and let a be an arbitrary element of S. Then $u \neq w$ and $u \in M(e) \vee w \in M(e)$. Thus, the first, we have $u \neq a \vee a \neq w$, i.e. we have $(u,w) \neq (a,a)$ what means that $t(e)$ is a consistent relation. The second, let v be an arbitrary element of S. Then $u \neq v \vee v \neq w$ and $u \in M(e) \vee w \in M(e)$. We have, for example,

$$\begin{aligned} u \neq v \wedge w \in M(e) &\Rightarrow u \neq v \wedge (w \in M(e) \wedge (w \neq v \vee v \in M(e))) \\ &\Rightarrow (u \neq v \wedge (w \in M(e) \wedge w \neq v)) \vee (u \neq v \wedge (w \in M(e) \\ &\quad \wedge v \in M(e))) \\ &\Rightarrow (v,w) \in t(e) \vee (u,v) \in t(e); \end{aligned}$$

In the case $u \neq v \wedge u \in M(e)$ we have simply $(u,v) \in t(e)$. Analogously, we have the implications $v \neq w \wedge u \in M(e) \Rightarrow (v,w) \in t(e) \vee (u,v) \in t(e)$ and $v \neq w \wedge w \in M(e) \Rightarrow (v,w) \in t(e)$.

So, the relation $t(e)$ is cotransitive. It is clear that $t(e)$ is a symmetric relation.
(2) The implication (*) is clear. For the proof of the implication (**) let we get $a \in G_e$ and let b in $at(e)$. Then $a \neq b \wedge b \in M(e)$ because $G_e \cap M(e) = \emptyset$. So, $at(e) \subseteq M(e)$. Beside that, for $x \in M(e) \subseteq \overline{G_e}$ we have $x \neq a$. So, $(a,x) \in t(e)$ and $x \in at(e)$. Therefore $at(e) = M(e)$. □

Corollary 3.3. *Let e be an idempotent of a semigroup S with apartness. Then the relation $t(e)^* = \{(x,y) \in S \times S \ :\ (\exists a,b \in S)(axb \neq ayb \wedge (axb \in M(e) \vee ayb \in M(e)) \in t(e))\}$ is a coequality relation on S compatible with the semigroup operation on S.*

Proof. See Corollary 1.7.2 in [**5**]. □

Corollary 3.4. *Let e be an idempotent of a semigroup S with apartness such that the maximal sugroup G_e is detachable in S. Then the relation $t(e)$ has the family of classes $\mathbf{V}(S,t(e)) = \{\{a \in S \ :\ a \neq x\} \mid x \in G_e\} \cup M(e)$.*

Proof. Let x be an element of S. Then $x \in G_e$ or $x \# G_e$. Therefore, if $x \in G_e$, then, by Corollary 3.2., we have $xt(e) = M(e)$. Let $x \# G_e = \overline{M(e)}$. Then $x \in M(e)$ and $xt(e) = \{a \in S \ :\ a \neq x\}$. □

Corollary 3.5. *Let e be an idempotent of a semigroup S with apartness such that the cosubsemigroup $M(e)$ is a coideal of S. Then the relation $t(e)$ is a cocongruence on S.*

Proof. Let $(ax, by) \in t(e)$, i.e. let $ax \neq by$ and $ax \in M(e) \vee by \in M(e)$. Then $a \neq b \vee x \neq y$ and $(a \in M(e) \wedge x \in M(e)) \vee (b \in M(e) \wedge y \in M(e))$. Therefore, $(a, b) \in t(e)$ or $(x, y) \in t(e)$. $\square$

Corollary 3.6. *Let e and f be idempotents of a semigroup S with apartness. Then there exists a strongly extensional and embedding function $\varphi : S \to \mathbf{V}(S, t(e)) \times \mathbf{V}(S, t(f))$ such that $(\pi_e \cdot \varphi)(S) = \mathbf{V}(S, t(e))$ and $(\pi_f \cdot \varphi)(S) = \mathbf{V}(S, t(f))$.*

Proof. Let a and b be elements of a semigroup S with idempotents $e, f \in E(S)(\neq \{1\})$ such that $a \neq b$. Then from $S = M(e) \cup M(f)$ we conclude that $a \in M(e) \vee a \in M(f)$ and $b \in M(e) \vee b \in M(f)$. Therefore, there exist coequality relations $t(e)$ and $t(f)$ such that $(a, b) \in t(e)$ or $(a, b) \in t(f)$. By Theorem 1.8. in [**5**], there exists a strongly extensional and embedding function $\varphi : S \to \mathbf{V}(S, t(e)) \times \mathbf{V}(S, t(f))$ such that $(\pi_e \cdot \varphi)(S) = \mathbf{V}(S, t(e))$ and $(\pi_f \cdot \varphi)(S) = \mathbf{V}(S, t(f))$. $\square$

Let S, K and Q be semigroups. Then S is a subdirect product of K and Q if there exists a strongly extensional and embedding homomorphims $\varphi : S \to K \times Q$ such that $\pi_K \cdot \varphi(S) = K$ and $\pi_Q \cdot \varphi(S) = Q$.

Corollary 3.7. *Let e and f be idempotents of a semigroup S with apartness such that $M(e)$ and $M(f)$ are coideals of S. Then S is subdirect product of semigroups $\mathbf{V}(S, t(e))$ and $\mathbf{V}(S, t(f))$.*

Proof. Let $M(e)$ and $M(f)$ be coideals of semigroup S. Then, by Corollary 3.5 in this paper, the coequality relations $t(e)$ and $t(f)$ are cocongruences on S and, by Corollary 1.7.1. in [**5**], the sets $\mathbf{V}(S, t(e))$ and $\mathbf{V}(S, t(f))$ are semigroups. Thus, by Corollary 3.6 of this paper, S is a subdirect product of semigroups $\mathbf{V}(S, t(e))$ and $\mathbf{V}(S, t(f))$. $\square$

Note: Let e and f be idempotents of a semigroup S such that $M(e)$ and $M(f)$ are coideals of S. Then $\mathbf{V}(S, t(e))$ and $\mathbf{V}(S, t(f))$ are semigroups and the sets $\mathbf{V}(S, t(e)) \times \{M(f)\}$ and $\{M(e)\} \times \mathbf{V}(S, t(f))$ are ideals of $\mathbf{V}(S, t(e)) \times \mathbf{V}(S, t(f))$. Let $\alpha : \mathbf{V}(S, t(e)) \times \{M(f)\} \to \mathbf{V}(S, t(e))$ and $\beta : \{M(e)\} \times \mathbf{V}(S, t(f)) \to \mathbf{V}(S, t(f))$ be strongly extensional and embedding bijections. Then we have the functions $E = \alpha^{-1} \cdot \pi_e \cdot \varphi \;:\; S \to \mathbf{V}(S, t(e)) \times \{M(f)\}$ and $F = \beta^{-1} \cdot \pi_f \cdot \varphi \;:\; S \to \{M(e)\} \times \mathbf{V}(S, t(f))$ such that $E(a) = (\pi_e \cdot \varphi(a), M(f))$ and $F(a) = (M(e), \pi_f \cdot \varphi(a))$ (for every a in S). Besides, the relation $q_e = \{(a, b) \in S \times S : E(a) \neq E(b)\}$ and the relation $q_f = \{(a, b) \in S \times S : F(a) \neq F(b)\}$ are coequality relations on S. As

$$\begin{aligned}
(a, b) \in t(e) &\Leftrightarrow a \neq b \wedge (a \in M(e) \vee b \in M(e)) \\
&\Rightarrow \varphi(a) \neq \varphi(b) \wedge at(e) = \{x \in S \;:\; a \neq x\},\, b \in at(e) \\
&\quad\; \wedge bt(e) = \{y \in S \;:\; b \neq y\},\, a \in bt(e) \\
&\Rightarrow \pi_e \cdot \varphi(a) = at(e) \neq bt(e) = \pi_e \cdot \varphi(b) \\
&\Leftrightarrow (\pi_e \cdot \varphi(a), M(f)) \neq (\pi_e \cdot \varphi(b), M(f)) \\
&\Leftrightarrow E(a) \neq E(b) \\
&\Leftrightarrow (a, b) \in q_e.
\end{aligned}$$

and similarly $t(f) \subseteq q_f$, we have, by Theorem 1.4 in [**5**], that the relations $t(e)/q_e$ and $t(f)/q_f$ are coequality relations on $\mathbf{V}(S, t(e))$ and $\mathbf{V}(S, t(f))$ respectively. Beside this, by Corollary 1.6.1 in [**5**], there exist strongly extensional bijective and embedding functions $\mathbf{V}(\mathbf{V}(S, q_e), t(e)/q_e) \to \mathbf{V}(S, t(e))$ and $\mathbf{V}(\mathbf{V}(S, q_f), t(f)/q_f) \to \mathbf{V}(S, t(f))$.

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Faculty of Sciences,
Department of Mathematics and Informatics,
Mladena Stojanovica 2, 78000 Banja Luka,
Republic of Srpska - Bosnia and Herzegovina
daniel@urcbl.bl.ac.yu

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