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# TWO REPRESENTATION OF REFLEXIVE $G$-INVERSES AND THEIR IMPLEMENTATION

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**In this paper we describe algorithms in the package for implementation of two methods for computing reflexive $g$-inverses. These methods are based on the following general solution of the matrix equations (1) and (2): $G = W_1(QW_1)^{-1}(W_2P)^{-1}W_2$. In the first algorithm we investigate implementation of a general determinantal representation for generalized inverses, which is introduced in [19]. These algorithms are continuation of the analogous algorithms developed in [19], written in the programming language . The second algorithm describes implementation of a modification of the hyper-power iterative method, introduced in [21].**

## 1. INTRODUCTION

Let $\mathbf{C}$ (resp. $\mathbf{R}$) be the field of complex (resp. real) numbers and $\mathbf{C}_r^{m\times n}$ (resp. $\mathbf{R}_r^{m\times n}$) be the set of $m\times n$ complex (real) matrices whose rank is $r$. Conjugate, transposed and conjugate-transposed matrix of $A$ are denoted by $\overline{A}$, $A^T$ and $A^*$, respectively. The determinant of a square matrix $B$ is denoted by $|B|$, and $\mathrm{Tr}(A)$ denotes the trace of $A$.

For a given $m\times n$ matrix $A$ over $\mathbf{C}$, let $\alpha=\{\alpha_1,\dots,\alpha_r\}$ and $\beta=\{\beta_1,\dots,\beta_r\}$ be subsets of $\{1,\dots,m\}$ and $\{1,\dots,n\}$, respectively. Then $|A_\beta^\alpha|$ denotes the minor of $A$ determined by the rows indexed by $\alpha$ and the columns indexed by $\beta$.

We use the following notation from [**10**]. For $1\le k\le n$, denote the collection of strictly increasing sequences of $k$ integers chosen from $\{1,\dots,n\}$, by

$$\mathcal{Q}_{k,n}=\{\alpha:\ \alpha=(\alpha_1,\dots,\alpha_k),\ \ 1\le\alpha_1<\cdots<\alpha_k\le n\}\,.$$

Let $\mathcal{N}=\mathcal{Q}_{r,m}\times\mathcal{Q}_{r,n}$. For fixed $\alpha\in\mathcal{Q}_{k,m}$, $\beta\in\mathcal{Q}_{k,n}$, $1\le k\le r$, let

$$\mathcal{I}(\alpha)=\{I:\ I\in\mathcal{Q}_{r,m},\ I\supseteq\alpha\}\,,\quad \mathcal{J}(\beta)=\{J:\ J\in\mathcal{Q}_{r,n},\ J\supseteq\beta\}\,,$$

$$\mathcal{N}(\alpha,\beta)=\mathcal{I}(\alpha)\times\mathcal{J}(\beta).$$

If $A$ is a square matrix, then the coefficient of $a_{ij}$ in the LAPLACE's expansion of $|A|$ is denoted by $\frac{\partial}{\partial a_{ij}}|A|$.

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For any matrix $A \in \mathbf{C}^{m\times n}$ the Moore-Penrose inverse of $A$ is the unique matrix, denoted by $A^\dagger$, satisfying the following Penrose's equations in $X$:

$$(1)\quad AXA=A,\quad (2)\quad XAX=X,\quad (3)\quad (AX)^*=AX,\quad (4)\quad (XA)^*=XA$$

and if $m = n$, also

$$(5)\qquad AX = XA.$$

For a sequence $\mathcal{S}$ of $\{1,2,3,4,5\}$, the set of matrices obeying the conditions contained in $\mathcal{S}$ is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an $\mathcal{S}$-inverse of $A$ and denoted by $A^{(\mathcal{S})}$. In the case $m = n$, the group inverse of $A$, denoted by $A^\#$, is the unique $\{1,2,5\}$ inverse of $A$.

Main properties of the weighted Moore-Penrose inverse are investigated in [**5**], [**13**]. By $A^\dagger_{M,N}$ we denote the unique solution of the equations (1), (2) and the following equations:

$$(6)\qquad (MAX)^* = MAX \qquad\qquad (7)\qquad (NXA)^* = NXA.$$

The methods implemented in this paper are based on the general representations of different classes of pseudoinverses, investigated in [**5**], [**7**], [**13**], [**15**], [**18**].

The paper is organized as follows. The second section contains implementation of the determinantal representation of generalized inverses, considered in [**4**], [**19**], [**20**], [**22**], so called the *general determinantal representation*. This implementation represents a continuation of the paper [**19**], where a general determinantal representation for the class of $\{1,2\}$ inverses is introduced. Also, in [**19**] are developed algorithms in the programming language `C` for implementation of the general determinantal representation.

In the third section we describe implementation in **MATHEMATICA** of an iterative method for computing $\{1,2\}$ inverses. This method is introduced in [**21**], and it is based on a generalization of the hyper-power method.

Several illustrative examples are given in the last section.

In this way, we obtain an extension of the programming system **MATHEMATICA**, by means of the implemented functions for computing the rank and index of a given matrix, and by means of the functions for computing the following classes of pseudoinverses: Moore-Penrose, weighted Moore-Penrose inverse, group inverse, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,2\}$ inverses, left/right inverses, Radić's and Stojaković's (Joshi's) generalized inverses. It is well known that in **MATHEMATICA** is available only the function *PseudoInverse* for computing the Moore-Penrose inverse [**25**], [**26**].

## 2. IMPLEMETATION OF GENERAL DETERMINTAL REPRESENTATION

For the sake of completeness, we restate here the general determinantal representation from the articles [**19**], [**20**]. This representation can be obtained from $G = W_1(QW_1)^{-1}(W_2P)^{-1}W_2$, where $A = PQ$ is an arbitrary full-rank factorization of $A$.

**Proposition 2.1.** *Let $A \in \mathbf{C}_r^{m\times n}$, and $A = PQ$ be its full-rank factorization. Then an arbitrary $\{1,2\}$ inverse $G = (g_{ij})$ of $A$ can be represented by the following determinantal representation*:

$$g_{ij} = \frac{\displaystyle\sum_{(\alpha,\beta)\in\mathcal{N}(j,i)} |(W_1W_2)^{T}{}_{\beta}^{\alpha}| \frac{\partial}{\partial a_{ji}} |A_\beta^\alpha|}{\displaystyle\sum_{(\gamma,\delta)\in\mathcal{N}} |(W_1W_2)^{T}{}_{\delta}^{\gamma}|\, |A_\delta^\gamma|}, \quad \left(\begin{smallmatrix} 1\le i\le n \\ 1\le j\le m\end{smallmatrix}\right),$$

*where $W_1 \in \mathbf{C}^{n\times r}$ and $W_2 \in \mathbf{C}^{r\times m}$ satisfy the conditions* (1.1)*.*

In [**19**] we introduce notions of the generalized determinant and the *general determinantal representation of different orders*. For the sake of clarity we introduce several notations.

The set $\mathcal{Q}_{t,m} \times \mathcal{Q}_{t,n}$, $t \le r = \operatorname{rank}(A)$ is denoted by $\mathcal{N}(t)$. For as given $m \times n$ complex matrix $R$, the *generalized determinant of the order* $t$, denoted by $\mathrm{DET}_{(R,t)}(A)$, can be expressed in this way (see [**19**]):

$$\mathrm{DET}_{(R,t)}(A) = \sum_{(\gamma,\delta)\in\mathcal{N}(t)} |\overline{R}_\delta^\gamma|\,|A_\delta^\gamma|. \tag{2.1}$$

Also, we introduce the following notation:

$$\mathcal{I}(\alpha,t) = \{I:\ I\in\mathcal{Q}_{t,m},\ I\supseteq\alpha\}, \quad \mathcal{J}(\beta,t) = \{J:\ J\in\mathcal{Q}_{t,n},\ J\supseteq\beta\},$$

$$\mathcal{N}(\alpha,\beta,t) = \mathcal{I}(\alpha,t)\times\mathcal{J}(\beta,t), \quad t\le r.$$

Then the *general determinantal representation of the order* $t$ for $A \in \mathbf{C}_r^{m\times n}$, can be written as follows (see [**19**]):

$$g_{ij} = a_{ij}^{(\dagger,R,t)} = \frac{\displaystyle\sum_{(\alpha,\beta)\in\mathcal{N}(j,i,t)} |\overline{R}_\beta^\alpha| \frac{\partial}{\partial a_{ji}} |A_\beta^\alpha|}{\mathrm{DET}_{(R,t)}(A)}, \quad \left(\begin{smallmatrix} 1\le i\le n \\ 1\le j\le m\end{smallmatrix}\right). \tag{2.2}$$

In the case $t = r$, we obtain well known result from [**4**]. Definitions of the generalized determinant and the general determinantal representation of different orders are useful in the case when the rank of a given matrix $A$ is unknown. Then we start the computation using (2.2) with $t = \min\{m,n\}$. Then we decrease values for $t$, until a nonzero value of $t$ satisfying $\mathrm{DET}_{(R,t)}(A) \ne 0$ is reached.

The general determinantal representation contains all known determinantal representations of generalized inverses, introduced in the papers [**3**–**6**], [**8**], [**12**–**13**], [**16**–**17**], [**23**–**24**].

A few connections between the general determinantal representation and the corresponding results from [**10**], [**14**] are investigated in [**22**].

Now, we describe implementation of the general determinantal representation of different orders, in the package MATHEMATICA. We begin by several auxiliary procedures.

**2.1.** By means of the following routine can be detected square matrices.

```
SquareMatrixQ[a_]:=Length[a]==Length[a[[1]]] /; MatrixQ[a]
```

**2.2.** The rank of any given matrix $A$ is equal to the number of nonzero elements in the reduced row echelon form of $A$. The result of the expression $zeros[u]$ is 0 if the vector $u$ is identical to the corresponding zero vector, and 1 otherwise.

```
zeros[u_]:=
   Block[{v=u,n,i=1,lg=0},
        n=Length[v];
        While[i<=n,
              If[v[[i]] =!= 0, lg=1];
              i++;
         ];
        lg
   ];
```

The function $rank[a]$ is a counter of all nonzero rows contained in the reduced row echelon form of $A$.

```
rank[a_]:=
   Block[{b=a,i,m,n,r,c},
         {m,n}=Dimensions[b];
         b=RowReduce[b];
         r=Sum[zeros[b[[i]]], {i,m}]
   ]; MatrixQ[a]
```

The index of a square matrix $A$ is defined as the first integer $k$ satisfying $\mathrm{rank}(A^{k+1})=\mathrm{rank}A^k$.

```
Index[a_]:=
  Block[{b=a,c=IdentityMatrix[Length[a]],d=a,k=0},
         While[Rank[c]=!=Rank[d],
                     d=d.b; c=c.b; k+=1
         ];
         k
  ] /; SquareMatrixQ[a]
```

**2.3.** The generalized determinant of the order $t$, defined by $\mathrm{DET}_{(R,t)}(A)$ in (2.1), can be computed by means of the following procedure *GDetR*. The formal parameters $a$ and $r$ denote the matrices $A$ and $R$, respectively, and the parameter $t$ denotes the size of the selected minors.

```
GDetR[a_,r_, t_Integer]:=
  Block[{b=a, ra=r, f, s, k, l, ma,mc},
         ma=Minors[b,t]; mc=Minors[ra,t];
         {f,s}=Dimensions[ma];
         Sum[Conjugate[mc[[k,l]]] ma[[k,l]], {k,f}, {l,s}]
       ]/; MatrixQ[a] && MatrixQ[r] &&
          Dimensions[a]==Dimensions[r] && Rank[a]==Rank[r]
```

**2.4.** In order to implement the general determinantal representation, firstly we develop two useful functions. The first function generates the submatrix of a given matrix $A$, obtained by deleting its $i$-th row and $j$-th column.

```
MatrixComp[a_, i_Integer, j_Integer]:=
  Block[{b=a},
         b=Drop[b,{i,i}]
         b=Transpose[Drop[Transpose[b],{j,j}]];
       ]/; MatrixQ[a]
```

In the second function we generate the submatrix of $A$ determined by the rows $p_1,\dots,p_t$ and columns $q_1,\dots,q_t$.

```
Minor[a_, p_List, q_List, t_Integer]:=
  Block[{b=a,i,j, c},
         c=IdentityMatrix[r];
         For[i=1, i<=t, i++,
              For[j=1, j<=t, j++,
                       c[[i,j]]=b[[p[[i]],q[[j]]]]
            ]   ];
         c
       ]/; MatrixQ[a]
```

**2.5.** Using an algorithm from [**9**], the set of all combinations of the order $t$ of the set $\{1,\dots,n\}$ can be implemented by the following code:

```
While[j>=1,
        If[j>=1,
               For[i=t, i>=j, i--,
                  p[[i]]=p[[j]]+i-j+1; p1[[i]]=p[[i]]
     ]     ]     ];
```

**2.6.** Finally, in the procedure *RINVERSE* we implement the general determinantal representation of the order $t \le r$, given by (2.2). The formal parameters $a$ and $r$ represent the input matrices $A$ and $R$, respectively. Initial value for the order $t$ of selected minors is $t = \min\{m,n\}$. In the while cycle the value of $t$ is decreased until the conditions $\mathrm{DET}_{(R,t)}(A) \neq 0$ is satisfied.

```
RInverse[a_,r_]:=
Block[{b=a,ra=r,t,p,q, m,n, w,v, i,j,k, j1, p1,q1,
          pr,pr1, awv,mr,mrr, mc,s,inv, sw,am},
  inv=Transpose[b];  {m,n}=Dimensions[b];
  t=Min[m,n]; d=GDetR[b,ra,t];
  While[d==0,    d=GDetR[b,ra,t];    t-- ];
  p=q=Range[t];    p1=q1=q;
  For[v=1, v<=n, v++,
   For[w=1, w<=m, w++,
       s=0;
       If[t==m, j=1, j=m];
       While[j>=1,
           If[t==n, j1=1, j1=n];
           While[j1>=1,
               pr=pr1=1;
               While[pr<=t && p[[pr]]=!=w, pr++];
               While[pr1<=t && q[[pr1]]=!=v, pr1++];
               If[pr<=t && pr1<=t,
                   mr=Minor[b,p,q,t];
                   mrr=Minor[ra,p,q,t];
                   mc=Conjugate[Det[mrr]];
                   am=Det[MatrixComp[mr,pr,pr1]];
                   awv=(-1)^(pr+pr1) am mc,
                   awv=0
               ];
               s+=awv;
               If[q[[t]]==n, j1--, j1=t ];
               If[j1>=1,
                   For[i=t, i>=j1, i--,
                      q[[i]]=q[[j1]]+i-j1+1;
                      q1[[i]]=q[[i]]
               ]   ]
           ];
           q1=q=Range[t];
           If[p[[t]]==m, j--, j=t ];
           If[j>=1,
               For[i=t, i>=j, i--,
                     p[[i]]=p[[j]]+i-j+1; p1[[i]]=p[[i]]
           ]   ]
       ];
       inv[[v,w]]=s/d
       p=q=Range[t];    p1=q1=q
   ]  ];
  inv
]/; MatrixQ[a]
```

**Remark 2.2.** *Described algorithms in* MATHEMATICA *are simpler and more efficient with respect to the corresponding in* [**19**], *written in* C.

*Computation of generalized inverses by means of the general determinantal representation a is direct method, and does not use the Gaussian elimination.*

## 3. MODIFICATION OF THE HYPER-POWER METHOD

The hyper-power iterative method was originally devised by ALTMAN [**2**] for inverting of a nonsingular bounded operator in a BANACH space. In [**11**] the convergence of the same method is proved under the condition which is weaker than the one assumed in [**2**], and some better error estimates are derived. ZLOBEC in [**30**] defined two hyper-power iterative methods of an arbitrary high order $q \geq 2$.

In the paper [**21**] we adapt the hyper-power method to be valid for computing all of the reflexive $g$-inverses.

**Proposition 3.1.** (see [**21**]) *Let rank*$(A) = r \geq 2$, *and the matrices* $W_1 \in \mathbf{C}^{n\times r}$, $W_2 \in \mathbf{C}^{r\times m}$ *satisfy conditions* (1.1). *If* $q \geq 2$ *is an integer, then both of the following two iterative methods*:

$$Y_0 = Y_0' = \alpha(W_2AW_1)^*, \qquad 0 < \alpha \leq \frac{2}{\mathrm{Tr}((W_2AW_1)^*W_2AW_1)},$$

$$\left\{\begin{array}{l} T_k = I_r - Y_kW_2AW_1, \\ Y_{k+1} = (I_r + T_k + \ldots + T_k^{q-1})Y_k, \\ X_{k+1} = W_1Y_{k+1}W_2 \end{array}\right. \quad \left\{\begin{array}{l} T_k' = I_r - W_2AW_1Y_k', \\ Y_{k+1}' = Y_k'(I_r + T_k' + \ldots + {T_k'}^{q-1}), \\ X_{k+1}' = W_1Y_{k+1}'W_2 \quad k = 0, 1, \ldots \end{array}\right.$$

*generate the class of the reflexive $g$-inverses of $A$.*

Under the suitable conditions, we get iterative methods for computing $\{1, 2, 3\}$ or $\{1, 2, 4\}$ inverses, the MOORE-PENROSE inverse, weighted MOORE-PENROSE inverse or the group inverse of $A$ (see [**21**]).

**3.1.** Implementation of the modified hyper-power method is given in the following. In order to compute the value $\alpha = \frac{2}{\mathrm{Tr}((W_2AW_1)^*W_2AW_1)}$, we need a function for computing the trace of a square matrix. This function is not built-in in MATHEMATICA. For this purpose we can use the following one-liner idea from [**1**]:

```
trace[mat_?MatrixQ]:=
    Plus @@(IdentityMatrix[Length[mat]] mat // Flatten)
```

We recommend the following routine:

```
trace[a_]:=
  Block[{b=a, i},
         Sum[b[[i,i]], {i,Length[b]}]
  ]/; SquareMatrixQ[a]
```

**3.2.** Now, we give the following implementation of the modified hyper-power method. In the following procedure the parameters $a$, $w1$, $w2$ represent the matrices $A$, $W_1$, $W_2$, respectively. The parameter $q$ denotes the order of the hyper-power expansion, and *numit* denotes the number of iterations.

```
HyperPower[a_,w1_,w2_,q_,numit_]:=
  Block[{tk,tk1,b=a,wa=w1,wb=w2,e,alpha,x,y,c=wb.b.wa, ra,s,i,k=1},
          ra=rank[b];
          alpha=2/trace[Conjugate[Transpose[c]].c];
          y=alpha Conjugate[Transpose[c]];
          e=IdentityMatrix[ra];
          While[k<numit,
                  tk1=tk=e-y.c; s=e;
                  Do[s+=tk; tk=tk1.tk,{i,q-1}];
                  y=s.y; x=wa.y.wb; k+=1
          ];
          x
  ]
```

## 4. EXAMPLES

**Example 4.1.** Consider the test matrix $S_5$ from [**27**], in the case $a=1$, i.e. $S_5 = \begin{pmatrix} 2 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 1 & 2 \end{pmatrix}$. Its full-rank factorization is, for example:

$$P = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

For the matrices $W_1$ and $W_2$ we can select, for example

$$W_1 = \begin{pmatrix} 1 & 2 & 5 & 3 \\ -2 & 4 & 0 & 3 \\ 2 & 1 & 0 & -2 \\ 0 & 5 & 0 & 1 \\ 7 & 2 & -3 & 2 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 2 & -2 & 1 & 1 & -5 \\ 3 & 0 & 1 & 4 & 0 \\ 0 & 2 & 1 & 3 & 4 \\ 7 & 1 & 1 & 9 & -3 \end{pmatrix}.$$

$RInverse[S_5,\ Transpose\ [W_1.W_2]]$ gives

$$S_5^{(1,2)} = \begin{pmatrix} \frac{28759}{10220} & -\frac{113}{140} & -\frac{1}{28} & \frac{151}{20} & -\frac{61317}{10220} \\ \frac{472}{73} & -2 & 1 & -1 & -\frac{399}{73} \\ -\frac{300}{73} & 1 & 0 & 1 & \frac{227}{73} \\ \frac{186}{73} & -1 & 1 & -2 & -\frac{113}{73} \\ -\frac{46539}{10220} & \frac{253}{140} & -\frac{27}{28} & -\frac{131}{0} & \frac{79097}{10220} \end{pmatrix}.$$

$RInverse[S_5,\ P.Transpose\ [W_1]]$ gives

$$S_5^{(1,2,3)} = \begin{pmatrix} \frac{223}{140} & -\frac{113}{140} & -\frac{1}{28} & \frac{151}{20} & -\frac{123}{140} \\ \frac{1}{2} & -2 & 1 & -1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & 1 & -\frac{1}{2} \\ \frac{1}{2} & -1 & 1 & -2 & -\frac{1}{2} \\ -\frac{223}{140} & \frac{253}{140} & -\frac{27}{28} & -\frac{131}{20} & \frac{223}{140} \end{pmatrix}.$$

$RInverse[S_5,\ Transpose\ [W_2].Q]$ gives

$$S_5^{(1,2,4)} = \begin{pmatrix} -\frac{127}{146} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{127}{146} \\ \frac{472}{73} & -2 & 1 & -1 & -\frac{399}{73} \\ -\frac{300}{73} & 1 & 0 & 1 & \frac{227}{73} \\ \frac{186}{73} & -1 & 1 & -2 & -\frac{113}{73} \\ -\frac{127}{146} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{127}{14} \end{pmatrix}.$$

The value of the expression $RInverse[\ S_5,\ S_5\ ]$ is

$$S_5^{\dagger} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -2 & 1 & -1 & \frac{1}{2} \\ -\frac{1}{2} & 1 & 0 & 1 & -\frac{1}{2} \\ \frac{1}{2} & -1 & 1 & -2 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

which is well-known result in **[27]**.

**Example 4.2.** Consider the matrix $A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$. Its full-rank factorization is $P = A$, $Q = I_2$. If we select $W_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $W_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, application of the modified hyper-power method of the order 2 leads to:

$$X_1 = \begin{pmatrix} 0 & -\frac{4}{49} & 0 \\ \frac{4}{49} & \frac{4}{49} & \frac{4}{49} \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & -\frac{376}{2401} & 0 \\ \frac{376}{2401} & \frac{376}{2401} & \frac{376}{2401} \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 0 & -\frac{1664176}{5764801} & 0 \\ \frac{1664176}{5764801} & \frac{1664176}{5764801} & \frac{1664176}{5764801} \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & -\frac{16417805178976}{33232930569601} & 0 \\ \frac{16417805178976}{33232930569601} & \frac{16417805178976}{33232930569601} & \frac{16417805178976}{33232930569601} \end{pmatrix},$$

$$X_5 = \begin{pmatrix} 0 & -x_5 & 0 \\ x_5 & x_5 & x_5 \end{pmatrix}, \; x_5 = \frac{821679232341479087467408576}{1104427674243920646305299201}.$$

We have obtained sequence converging to $X = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \in A\{1,2\}$. The matrix $A$ is of full column rank, so that $A^{(1,2)} = A^{(1,2,4)}$ [**29**], and consequently $X \in A\{1,2,4\}$.

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