UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. 10 (1999), 3–8.

ON MINIMAL SPACE CURVES IN THE SENSE OF BERTRAND CURVES

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In this work, it is shown that when a minimal space curve (C) is given, a minimal space curve (C^*) can be determined so that at corresponding points the curves have parallel tangents in the opposite directions and the distance between these points is constant. Defined curve pairs looks like Bertrand curves because of the common principal normals.

PRELIMINARIES

We give the basic concepts:

Definition 1. *Let x_p be a complex analytic function of a complex variable t. Then the vector function*

$$\vec{x}(t)=\sum_{p=1}^{3} x_p(t)\vec{k}_p,\quad t=t_1+it_2$$

is called an imaginary curve, where $\vec{x}:\mathbf{C}\to\mathbf{C}^3$ and $\vec{k}_p$ is a unit vector [**3**], [**4**].

As in [**3**], [**4**] according to the standard Eucledean metric $\mathrm{d}s^2=\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_3^2$.

Definition 2. *The curves, of which the square of the distance between the two points equal to zero, are called minimal or isotropic curves.*

Definition 3. *Let s denote the arc-length (see* [**3**], [**4**]*). A curve is a minimal curve if and only if $ds^2=0$.*

Let $\vec{x}=\vec{x}(t)$ be a minimal curve in space with t complex variable. Then from Definition 3 and Definition 2

$$\mathrm{d}s^2=\mathrm{d}\vec{x}^2=0.$$

1991 Mathematics Subject Classification: 53A04

For every regular point, it means that

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{x}'(t) \neq \vec{0}.$$

Isotropic curves in space $\vec{x} = \vec{x}(t)$ satisfy the differential equation

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{x}'(t) \neq \vec{0}, \quad [\vec{x}'(t)]^2 = 0.$$

Consequently minimal curves in space satisfy the following differential equation

$$[\vec{x}'(t)]^2 = 0.$$

By differentiation

$$\vec{x}'(t)\vec{x}''(t) = 0.$$

By trivial vector calculus we have

$$[\vec{x}'(t) \wedge \vec{x}''(t)]^2 = 0.$$

It is also an isotropic vector which is perpendicular to itself. Then

$$\vec{x}'(t) \wedge \vec{x}''(t) = \lambda \vec{x}'(t) \quad \lambda \neq 0$$

can be written. By vector product with $\vec{x}''(t)$ we find

$$\lambda^2 = -[\vec{x}'']^2.$$

Then by substitution we obtain

$$\vec{x}' = \frac{\vec{x}' \wedge \vec{x}''}{\sqrt{-x''^2}}.$$

For another complex variable t^*, $t = f(t^*)$, $\dfrac{\mathrm{d}f}{\mathrm{d}t^*} = f^{\cdot} \neq 0$

$$\mathrm{d}\vec{x} = \frac{\vec{x}' \wedge \vec{x}''}{\sqrt{-x''^2}}\,\mathrm{d}t = \frac{\vec{x}^{\cdot} \wedge \vec{x}^{\cdot\cdot}}{\sqrt{-x^{\cdot\cdot 2}}}\,\mathrm{d}t^*$$

where $\vec{x}^{\cdot} = \vec{x}' f^{\cdot}$, $\vec{x}^{\cdot\cdot} = \vec{x}'' f^{\cdot 2} + \vec{x}' f^{\cdot\cdot}$.

The equality

$$\vec{x}^{\cdot\cdot 2} = (\vec{x}'')^2 f^{\cdot 4}$$

can be written in the following form

$$(-x^{\cdot\cdot 2})^{\frac{1}{4}}\,\mathrm{d}t^* = (-\vec{x}''^2)^{\frac{1}{4}}\,\mathrm{d}t$$

If we choose t^* such that $\vec{x}^{\cdot\cdot 2} = -1$, then by integration

$$t^* = s = \int_{t_0}^{t} (-\vec{x}''^2)^{\frac{1}{4}}\,\mathrm{d}t$$

is obtained. It is called the pseudo arc-length of the minimal curve which is invariant with respect to the parameter t (see [**4**]).

For each point $N(s)$ of the minimal curve, an E. Cartan frame field is defined, as follows (see [**1**], [**4**]).

(1)
$$\begin{aligned}
&\vec{e}_1 = \vec{x}^{\cdot}\\
&\vec{e}_2 = i\vec{x}^{\cdot\cdot}\\
&\vec{e}_3 = -\tfrac{\beta}{2}\vec{x}^{\cdot} + \vec{x}^{\cdot\cdot}, && \beta = \vec{x}^{\cdot\cdot\cdot 2}\\
&\vec{e}_i\cdot\vec{e}_j = \begin{cases} 1 & i+j=4\\ 0 & i+j\neq 4\end{cases}\\
&\vec{e}_i\wedge\vec{e}_j = (\vec{e}_{i+j-2})i && (i,j=1,2,3),\ (\vec{e}_1,\vec{e}_2,\vec{e}_3)=i
\end{aligned}$$

$s = \int_{t_0}^{t}(-\vec{x}''^2)^{\frac{1}{4}}\,\mathrm{d}t$ is a pseudo arc-length also invariant with respect to the parameter t. If we differentiate (1) with respect to the s, which is a pseudo arc-length of the minimal curve, the following equations can be deduced:

(2)
$$\vec{e}_1^{\,\cdot} = -i\vec{e}_2,\ \vec{e}_2^{\,\cdot} = i(k\vec{e}_1+\vec{e}_3),\ \vec{e}_3^{\,\cdot} = -ik\vec{e}_2,$$

where $k = \dfrac{\beta}{2}$ is called pseudo-curvature.

These equations can be used if the minimal curve is at least of class C^4. In the solution of the problem of our paper we used also the method of [2].

MINIMAL SPACE CURVES IN THE SENSE OF BERTRAND CURVES

Let (C) and (C^*) be a pair of minimal space curves of class C^4 with non-vanishing pseudo curvature in Eucledian space. In contrast to the Bertrand curves we shall assume that our curves have parallel tangents in opposite directions with common principal normals, at corresponding points. According to the (1) the square of the arc-length is $\mathrm{d}s^2 = \mathrm{d}x_2^2 + \mathrm{d}x_1\,\mathrm{d}x_3$. We can write for the equation of the curve (C^*):

(3)
$$\vec{\alpha}^* = \vec{\alpha}(s) + \lambda\vec{e_1} + \mu\vec{e_2} + \delta\vec{e_3}$$

Differentiating this equation with respect to s, which is an arc-length of (C), and using the Cartan formulas (2), we obtain,

$$\frac{\mathrm{d}\vec{\alpha}^*}{\mathrm{d}s} = \vec{e}_1^{\,*}\frac{\mathrm{d}s^*}{\mathrm{d}s} = \vec{e}(1-ki\lambda^{\cdot}+\mu ik) + \vec{e}_2(-\lambda i - ki\mu^{\cdot} - ik\delta) + \vec{e}_3(\mu i - ki\delta^{\cdot})$$

Since at the corresponding points of (C) and (C^*) $-\,\vec{e}_1^{\,*} = \vec{e}_1$ we have

(4)
$$1 - ki\lambda^{\cdot} + \mu ik = -\frac{\mathrm{d}s^*}{\mathrm{d}s},\ -\lambda i - ki\mu^{\cdot} - \delta ki = 0,\ \ \mu i - \delta^{\cdot} ki = 0.$$

Arc differential of $\vec{e}_3 = \vec{e}_3(s)$ is $\mathrm{d}\phi = \pm\sqrt{\left(\frac{\mathrm{d}\vec{e}_3}{\mathrm{d}s}\right)^2}\,\mathrm{d}s$.

Let us denote the radii of pseudo curvatures of (C) and (C^*) respectively by

$$\frac{\mathrm{d}\phi}{\mathrm{d}s} = -ki = \frac{1}{\rho}, \quad \frac{\mathrm{d}\phi}{\mathrm{d}s^*} = -k^* i = \frac{1}{\rho^*}. \tag{5}$$

Hence, using (5), (4) can be writen as

$$\mu = \lambda' + f(\phi),\ \lambda = \frac{\mu'}{\rho} i \cdot - \frac{\delta}{\rho} i,\ \delta' = -\mu i \rho, \tag{6}$$

where $f(\phi) = \rho + \rho^{*}\cdot$. Eliminating μ, δ and their derivatives from the equation (6), we get the following linear differential equation of the third order in λ:

$$\lambda''' - 2\rho i \lambda' - \rho i \lambda - i\rho f + f'' = 0, \tag{7}$$

where a dash denotes differentiation with respect to ϕ.

If the distance between corresponding points of (C) and (C^*) is constant, we may write (here the scalar product is defined as in (1))

$$\|\vec{\alpha}^* - \vec{\alpha}\|^2 = \|\vec{d}\|^2 = \mu^2 + 2\lambda\delta = \text{constant}. \tag{8}$$

Differentiation of (8) yields

$$\frac{1}{2}\frac{\mathrm{d}\|\vec{d}\|^2}{\mathrm{d}\phi} = \mu\mu' + \lambda'\delta + \lambda\delta' = 0. \tag{9}$$

By virtue of (6), (9) is reduced to

$$\delta f = 0. \tag{10}$$

Here, there are two main cases to consider: First, if $f(\phi) = 0$ then the vector $\vec{d}$ is constant. To verify this fact, differentiate

$$\vec{d} = \lambda\vec{e_1} + \mu\vec{e_2} + \delta\vec{e_3} \tag{11}$$

and use (6); $f(\phi) = 0$ implies $\frac{\mathrm{d}\vec{d}}{\mathrm{d}\phi} = \vec{0}$. Conversely, if $\frac{\mathrm{d}\vec{d}}{\mathrm{d}\phi} = \vec{0}$, then it follows from (6) $f(\phi) = 0$. So we can give the following theorems:

Theorem 1. *The distance between corresponding points of (C) and (C^*) is constant if and only if $f(\phi) = 0$.*

Theorem 2. *(C^*) is a translation of C by a constant vector if and only if $f(\phi) = 0$. In this case, (7) reduces to the following equation:*

$$\lambda''' - 2\rho i \lambda' - \rho' i \lambda = 0.$$

Theorem 3. *If* (C^*) *is a translation of* (C) *by a constant vector the parameter* λ *in equation* (3) *verifies the following differential equation.*

$$\lambda''' - 2\rho i\lambda' - \rho' i\lambda = 0. \tag{12}$$

Conclusion 1. *In addition to* $f(\phi) = 0$, *in the case of* $\mu = 0, \lambda =$ *constant, we have from* (6)

$$\frac{\lambda}{\delta} = \frac{l}{\rho i} = -k = \text{constant}.$$

This means (C) *is pseudo helix (see* [**4**]*).*

Now we shall discuss the second case: $\delta = 0$. The equation (6) becomes

$$\mu = \lambda' + f,\ \rho\lambda = \mu' i,\ -\mu i\rho = 0. \tag{13}$$

Conclusion 2. *If* $\rho = 0$, *from* (13) *we have* $\mu = 0$, $\lambda = 0$, $f(\phi) = 0$. *The equation* (3) *yields,*

$$\vec{\alpha}^* = \vec{\alpha}.$$

That is (C) *and* (C^*) *coincide.*

$\rho = 0$ implies $\mu = \text{constant}, \lambda' = A - f$, that is

$$\begin{aligned} \lambda &= \int_0^\phi A\,\mathrm{d}\phi - \int_0^\phi f(\phi)\,\mathrm{d}\phi, \\ \vec{\alpha}^* &= \vec{\alpha} + \left[A\phi - \int_0^\phi f(\phi)\,\mathrm{d}\phi\right]\vec{e}_1 + A\vec{e}_2, \\ \vec{d}^2 &= \|\vec{\alpha}^* - \vec{\alpha}\|^2 = A^2. \end{aligned}$$

Then we can give the following conclusion.

Conclusion 3. *When the minimal curve* (C) *is given, then there is an infinite number of minimal curves* (C^*). *The distance between the corresponding points of a pair of curves* (C) *and* (C^*) *is* $A =$ *constant.*

Let (C) be an isotrop cubic $(k = 0)$ (see [**4**]). From (4), we get

$$\mathrm{d}s = -\,\mathrm{d}s^*,\ \lambda = 0,\ \mu = 0. \tag{14}$$

Hence, the equation of (C^*) is given as follows:

$$\vec{\alpha}^* = \vec{\alpha} + \delta\vec{e}_3. \tag{15}$$

It is obvious that when the isotrop cubic is given, an infinite number of curves (C^*) can be derived, and the distance between the corresponding points of a pair of curves is

$$\|\vec{\alpha}^* - \vec{\alpha}\| = 0.$$

From the differentiation of (15) with respect to pseudo length of (C) and using the CARTAN formula (2) we deduce $k^* = 0$, where $\delta = A =$ constant. Therefore we can give the following theorem.

Theorem 4. *If the minimal curve (C) is an isotrop cubic under the condition $\delta = A =$ constant, C^* is also an isotrop cubic.*

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(Received November 28, 1995)
(Revised October 9, 1998)