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A DATABASE OF STAR COMPLEMENTS OF GRAPHS

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Let μ be an eigenvalue of the graph G with multiplicity k. A star complement for μ in G is an induced subgraph $H = G - X$ such that $|X| = k$ and μ is not an eigenvalue of $G - X$. The database contains about 1500 triples (G, H, μ) and is available as a supplement to the programming package "Graph". It was produced using (a) "Graph" itself, (b) programs developed independently by M. Lepović, and (c) data from other sources cited in the bibliography. This paper contains a description of the database and a commentary which explains how some interesting graphs can be obtained by extending appropriate star complements.

1. INTRODUCTION

We take G to be an undirected graph without loops or multiple edges, with vertex set $V(G) = \{1, \dots, n\}$, and with $(0,1)$-adjacency matrix A. Let P denote the orthogonal projection of $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}(\mu)$ of A, and let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be the standard orthonormal basis of $\mathbb{R}^n$. Since $\mathcal{E}(\mu)$ is spanned by the vectors $P\mathbf{e}_j$ $(j = 1, \dots, n)$ there exists $X \subseteq V(G)$ such that the vectors $P\mathbf{e}_j$ $(j \in X)$ form a basis for $\mathcal{E}(\mu)$. Such a subset X of $V(G)$ is called a *star set* for μ in G. (The terminology reflects the fact that the vectors $P\mathbf{e}_1, \dots, P\mathbf{e}_n$ form a eutactic star in the sense of Seidel [**23**]. In the context of star partitions [**10**, Section 7.1], star sets are called *star cells*.) An equivalent definition which is useful in a computational context is the following: if μ has multiplicity k then a star set for μ in G is a set X of k vertices of G such that μ is not an eigenvalue of $G - X$ [**10**, Theorem 7.2.9]. Here $G - X$ is the subgraph of G induced by $\overline{X}$, the complement of X in $V(G)$. Accordingly, the graph $G - X$ is called the *star complement* for μ corresponding to X. (Such graphs are called *μ-basic* subgraphs in [**15**].)

If a single vertex is deleted from a graph then by the Interlacing Theorem [**5**, Theorem 0.19] the multiplicity of an eigenvalue changes by 1 at most. Accordingly, the deletion of any r vertices from X $(0 < r < k)$ results in a graph for which μ is an eigenvalue of multiplicity $k - r$. We can also make the following observations: (i) if K is an induced subgraph of G not having μ as an eigenvalue then G has a star complement for μ containing K [**21**, Proposition 1.1], (ii) a connected graph has a connected star complement for each eigenvalue [**21**, Theorem 2.4]. (The second result is ascribed to S. Penrice in [**15**].)

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If X is a star set for the non-zero eigenvalue μ of G then $\overline{X}$ is a dominating set in G [**10**, Theorem 7.3.1]; moreover if $\mu \notin \{-1, 0\}$ then $\overline{X}$ is a location-dominating set, meaning that the $\overline{X}$-neighbourhoods of vertices in X are distinct and non-empty [**10**, Corollary 7.3.6]. It follows that if $\mu \notin \{-1, 0\}$ and $|\overline{X}| = t$ then $|X| < 2^t$; thus there are only finitely many graphs which have an eigenspace $\mathcal{E}(\mu)$ $(\mu \neq -1, 0)$ with prescribed codimension t. In particular there are only finitely many graphs with a prescribed star complement $H = G - X$ corresponding to a multiple eigenvalue μ other than -1 or 0. If we prescribe not only μ and $G - X$ but also the set $E(X, \overline{X})$ of edges between X and $\overline{X}$ then G is determined uniquely: this is the Reconstruction Theorem [**10**, Theorem 7.4.1] (see Theorem 1.1 below). Taken together these observations provide the basis for determining the graphs with a prescribed star complement H (cf. [**16**], [**17**]) and characterizing certain graphs by properties of H which have implications for $E(X, \overline{X})$ (cf. [**18**], [**19**]). For example, the complement of the SCHLÄFLI graph is the unique maximal graph with a star complement $K_{2,5}$ corresponding to a multiple eigenvalue other than -1 [**16**]. The graphs having a star complement with at most 5 vertices, for an eigenvalue other than -1 or 0, are determined in [**22**].

The eigenvalues -1 and 0 are necessary exceptions: if $\mu = -1$, X may contain arbitrarily many pairwise adjacent vertices with the same closed neighbourhood ([**21**, Proposition 2.1(ii)(c)]), while if $\mu = 0$, X may contain arbitrarily many independent vertices with the same open neighbourhood. This is because the rank of the adjacency matrix A (when $\mu = 0$) or $A + I$ (when $\mu = -1$) is independent of the number of these so-called *duplicate* vertices (cf. [**15**, Section 1]). If duplicate vertices are excluded then again only finitely many graphs arise for given $t = |\overline{X}|$: these are the graphs described by ELLINGHAM [**15**] as *reduced* $(\mu = 0)$ or *co-reduced* $(\mu = -1)$. Accordingly, the eigenvalues -1 and 0 afford no essential obstruction to star complement techniques.

We note that when $\mu \notin \{-1, 0\}$ (or when $\mu \in \{-1, 0\}$ and G has no duplicate vertices) the exponential bound $|X| < 2^t$ may be improved to the quadratic bound $|X| \leq (t-1)(t+4)/2$ [**20**]. Consideration of the line graph $L(K_t)$, for which $\mathcal{E}(-2)$ has codimension t, shows that an upper bound of order $t^2/2$ for $|X|$ is asymptotically best possible as $t \to \infty$.

The following fundamental result combines the Reconstruction Theorem [**10**, Theorem 7.4.1] with its converse [**10**, Theorem 7.4.4].

Theorem 1.1. *Let G be a graph with adjacency matrix $\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix}$. Then X is a star set for μ in G if and only if μ is not an eigenvalue of C and*

$$\mu I - A_X = B^T (\mu I - C)^{-1} B. \tag{1}$$

Note that if X is a star set for μ then the corresponding star complement $H (= G - X)$ has adjacency matrix C, and Equation (1) tells us that G is determined by μ, H and the H-neighbourhoods of vertices in X. In this situation, let $|\overline{X}| = t$ and define a bilinear form on $\mathbb{R}^t$ by $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T (\mu I - C)^{-1} \mathbf{y}$. Denote the columns of B by $\mathbf{b}_u$ $(u \in X)$.

To discuss graphs with a prescribed star complement we shall use the following consequence of Theorem 1.1.

Corollary 1.2. *Suppose that μ is not an eigenvalue of the graph H. There exists a graph G with a star set X for μ such that $G-X=H$ if and only if the characteristic vectors $\mathbf{b}_u$ $(u \in X)$ satisfy*

(i) $\langle \mathbf{b}_u, \mathbf{b}_u \rangle = \mu$ *for all $u \in X$, and*

(ii) $\langle \mathbf{b}_u, \mathbf{b}_v \rangle \in \{-1, 0\}$ *for all pairs u, v of distinct vertices in X.*

If G has H as a star complement for μ with corresponding star set X then each induced subgraph $G-Y$ $(Y \subseteq X)$ also has H as a star complement for μ. Moreover any graph with H as a star complement for μ is an induced subgraph of such a graph G for which X is maximal, because H-neighbourhoods determine adjacencies among vertices in a star set. Accordingly in determining all the graphs with H as a star complement for μ, it suffices to describe those for which a star set X is maximal. Let H have t vertices, and for $\Delta \subseteq V(H)$ let H_Δ denote the graph obtained from H by adding a vertex with Δ as its neighbourhood. For $\mu \in \mathbb{R}$ let $\mathcal{S}(H,\mu)$ be the set of those Δ for which μ is an eigenvalue of H_Δ but not of H. When $\mathcal{S}(H,\mu)$ is not empty it is convenient to define the *extendability graph* $\Gamma(H,\mu)$ on $\mathcal{S}(H,\mu)$ as follows (cf. [**15**, Algorithm 2.4]): Δ_1 is adjacent to Δ_2 in $\Gamma(H,\mu)$ if and only if Δ_1, Δ_2 feature as H-neighbourhoods in a $(t+2)$-vertex graph for which H is a star complement. If we identify a set Δ with its characteristic vector, the vertices of $\Gamma(H,\mu)$ are the $(0,1)$-vectors $\mathbf{b}$ in $\mathbb{R}^t$ such that $\langle \mathbf{b}, \mathbf{b} \rangle = \mu$, and $\mathbf{b}_1 \sim \mathbf{b}_2$ if and only if $\langle \mathbf{b}_1, \mathbf{b}_2 \rangle \in \{-1, 0\}$. A graph G with a maximal star set X for μ such that $G-X=H$ now corresponds to a maximal clique in $\Gamma(H,\mu)$.

Our database contains data which represent solutions or partial solutions to several instances of the following three problems.

Problem 1. *For a given graph G and for each eigenvalue μ of G, find all the star complements for μ and partition them into isomorphism classes and/or equivalence classes generated by the automorphism group of G.*

Problem 2. *For a given graph H (and, optionally, for a given real μ) find all maximal graphs having H as a star complement (for μ).*

More generally, Problem 2 can be put in the following form:

Problem 2′. *For a given class $\mathcal{H}$ of graphs (and, optionally, for a given real μ) find all maximal graphs having as star complements (for μ) the graphs from $\mathcal{H}$.*

Additional data on star complements of particular graphs can be found in [**1**], [**11**], [**12**], [**16**], [**20**], [**21**], [**22**].

2. A DESCRIPTION OF THE DATABASE

The database of star complements is implemented in the environment of the expert system "Graph", a programming package for graph theory [**6**], [**8**]. The system "Graph" has been used extensively to support research in graph theory, in particular the theory of graph spectra [**9**]. Without describing in detail the options provided by "Graph", one can say that they enable graphs from the database to be processed in many different ways. Standard graphs such as paths, stars, cycles and complete graphs, can be generated simply by specifying the number of vertices; and there are commands which enable all one-vertex extensions of a given graph to be constructed. Several graph invariants (including eigenvalues, eigenvectors, angles and main angles) can be calculated. Graphs can be displayed on a screen and modified using a mouse; for example it is easy to add or delete vertices or edges. Working interactively one can study extensions of star complements as well as finding star complements as induced subgraphs.

Graphs from the database are stored under given names in several files. Each file can be inspected by "Graph" and any command of "Graph" can be applied to all graphs in the file under consideration. Moreover it is possible to move graphs from one file to another.

Most graphs in the database were generated either by the system "Graph" or by an independent program (called "Star") developed by one of the authors (M. L.), and then converted to the format required for "Graph". For the exceptional graphs discussed in the next section, we are indebted to F. C. Bussemaker. These graphs were originally generated for the paper [**14**], and a description may be found there. The program "Star" generates all one-vertex extensions of a given graph and finds all eigenvalues of all extensions. For an eigenvalue selected by the user, the program constructs the extendability graph, finds the maximal cliques therein, and thereby identifies the maximal graphs and their isomorphism classes.

3. EXCEPTIONAL GRAPHS

A graph is called *exceptional* if its least eigenvalue is greater than or equal to -2 and it is not a generalized line graph (see [**10**], p. 5).

Connected exceptional graphs with least eigenvalue greater than -2 have $6, 7$ or 8 vertices [**14**]. There are 20 such graphs on 6 vertices, 110 on 7 and 443 on 8 vertices. It is proved in [**11**] that a connected exceptional graph has an exceptional star complement for the least eigenvalue -2. It follows that every exceptional graph is the extension of (at least) one of the aforementioned 573 exceptional graphs with least eigenvalue greater than -2. Accordingly, for each of these 573 graphs we have included in our database all one-vertex extensions with -2 as an eigenvalue, and all two-vertex extensions with -2 as a double eigenvalue. This information is sufficient for the construction of the corresponding extendability graphs as a means of determining all the exceptional graphs (an instance of Problem $2'$); see Example 5 in Section 4.

The 20 exceptional graphs on 6 vertices are included in the table of [**7**]. The smallest pair of cospectral connected graphs [**5**, p. 57], is among these 20 graphs. The 110 exceptional graphs on 7 vertices with least eigenvalue greater than -2 have the following identification numbers in the table of connected graphs on 7 vertices from the book [**4**]:

3	14	15	29	33	39	62	63	64	67
86	94	98	99	100	106	122	148	149	155
185	186	190	194	196	197	198	201	205	215
261	262	268	303	306	307	309	310	315	318
339	340	343	344	348	349	388	390	427	428
430	431	433	435	446	464	468	469	472	473
474	476	477	482	486	540	574	575	578	579
581	588	589	599	601	602	603	606	608	663
683	684	685	686	689	701	702	703	707	709
741	755	757	758	767	772	773	797	800	807
808	810	812	823	827	832	842	847	849	851

We note in passing that these include 8 pairs of cospectral exceptional graphs on 7 vertices:

$$62,63;\ 92,93;\ 98,99;\ 148,149;\ 197,198;\ 339,340;\ 343,344;\ 588,589.$$

The data for the 443 exceptional graphs on 8 vertices are published in [**3**]. They were generated by several researchers using a computer, and can also be generated by "Graph".

4. EXAMPLES AND COMMENTS

The database of star complements was created primarily as a means of investigating the extent to which graphs are characterized by star complements. This approach has been vindicated by the results which have been obtained (cf. [**1**],[**11**],[**12**]). Often the maximal graphs which arise as extensions of a prescribed star complement H include interesting graphs which are strongly regular (for example, the Clebsch graph [**17**] and the Schläfli graph [**16**]) or non-regular graphs with three distinct eigenvalues (see Examples 4 and 5).[1] In some cases such a graph may be associated with a design or an intersecting family of sets, namely the H-neighbourhoods of vertices outside H [**17**].

We now give some details which illustrate how the database has been used. We start with an example related to an instance of Problem 1.

Example 1. For the graph G in Fig. 1 and each eigenvalue of multiplicity 3 we give in the same figure the 11 star complements which arise.

[1] The existence of connected non-regular graphs with three distinct eigenvalues, other than complete bipartite graphs, was noted for the first time in [**2**] and such graphs have recently attracted the attention of other researchers (see, for example, [**13**, Chapter 2]).

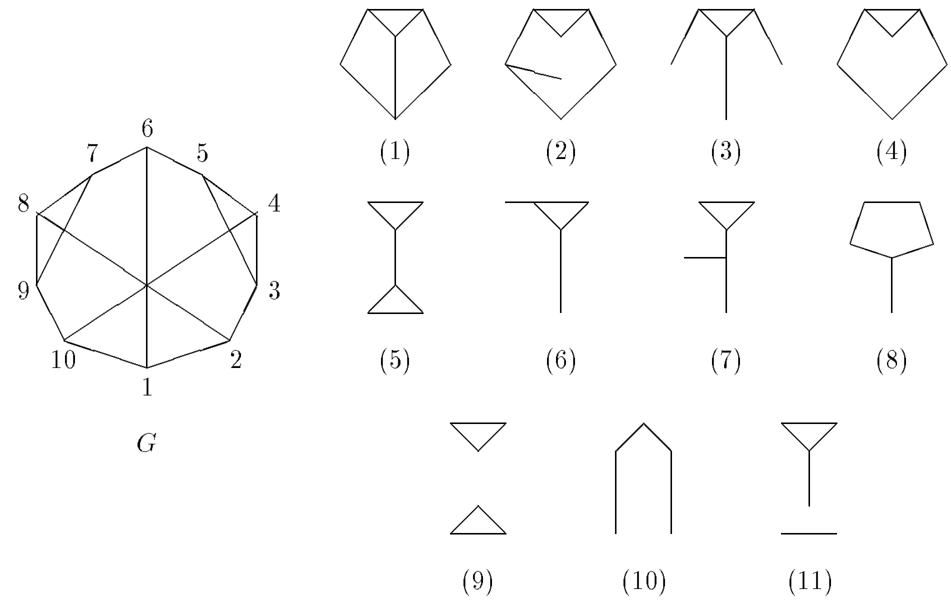

Fig. 1

The graph G has spectrum $3, 2, 1^3, -1^2, -2^3$. The star complements (1)–(10) are star complements for both the eigenvalue 1 and the eigenvalue -2, while (11) is a star complement only for the eigenvalue -2. The star complement (9) has a unique imbedding in G, while others have various (possibly isomorphic) imbeddings. In this example, it turns out that for isomorphic star complements, the subgraphs induced by the corresponding star cells are always isomorphic, but this statement is not true in general (see [**1**, Section 1]).

The star complements (1)–(11) are presented in Table 1, where column 1 contains the identification number, column 2 gives the largest eigenvalue, column 3 gives the number of mutually isomorphic star complements in each case, and column 4 gives the number of edges in the corresponding star cells. The star complements (5) and (6), which have the same largest eigenvalue (or *index*), are distinguished by the second largest eigenvalue (given in parentheses), but the star complements (3) and (4) are cospectral.

(1)	2.6554		2	3
(2)	2.4877		12	2
(3)	2.4383		6	1
(4)	2.4383		6	1
(5)	2.3429	(2.0000)	3	2
(6)	2.3429	(1.4142)	12	1
(7)	2.2970		12	1
(8)	2.1515		18	1
(9)	2.0000		1	0
(10)	1.8478		12	0
(11)	2.2143		6	0

Table 1

Non-isomorphic imbeddings of isomorphic star complements arise in case (8), as can be seen by considering the star cells $\{2, 4, 8\}$ and $\{4, 5, 8\}$. The subgraphs induced by these cells are isomorphic and contain one edge (edge $\{2, 8\}$ in the first case and $\{4, 5\}$ in the second). Although the corresponding star complements are isomorphic, the subgraphs obtained by deleting from G the edge $\{2, 8\}$ and the edge $\{4, 5\}$ are not isomorphic. (The first contains two triangles while the second contains only one triangle.) Further the bipartite graphs induced by the corresponding co-boundaries are not isomorphic.

The next three examples are related to instances of Problem 2.

Example 2. We find the connected graphs which have C_7 as a star complement for a multiple eigenvalue μ. We first inspect the spectra of the seventeen connected 8-vertex graphs $C_7 + u$ containing an induced 7-cycle. The eigenvalues of $C_7 + u$ are: $\lambda_1 > \lambda_2 \geq \lambda_3 > \lambda_4 \geq \lambda_5 > \lambda_6 \geq \lambda_7 > \lambda_8$, where $\lambda_3 = 1.2470$, $\lambda_5 = -0.4450$ and $\lambda_7 = -1.8019$ (these being the double eigenvalues of C_7). The remaining eigenvalues are given in Table 2, where the first column lists the neighbours of u as vertices of a 7-cycle 12345671. Since the index of a connected graph is simple we know that μ is not the index of a graph $C_7 + u$, hence not an algebraic conjugate of the index. It follows from Table 2 that $\mu \in \{-2, -1, 0\}$.

We make use of the Reconstruction Theorem in the form

$$m(\mu)(\mu I - A) = B^T m(\mu)(\mu I - C)^{-1} B, \tag{2}$$

where $m(x)$ is the minimal polynomial of $C = \mathrm{circul}(-8, 1, 0, 0, 0, 0, 1)$, i.e. $m(x) = x^4 - x^3 - 4x^2 + 3x + 2$. We define $\langle\langle \mathbf{x}, \mathbf{y} \rangle\rangle = \mathbf{x}^T m(\mu)(\mu I - C)^{-1}\mathbf{y}$ $(\mathbf{x}, \mathbf{y} \in \mathrm{IR}^7)$. The vertices of the corresponding graph $\Gamma(C_7, \mu)$ are the $(-8, 1)$-vectors $\mathbf{b} \in \mathrm{IR}^7$ such that $\langle\langle \mathbf{b}, \mathbf{b} \rangle\rangle = \mu m(\mu)$, and these are obtained from the graphs which have an eigenvalue μ in Table 2.

If $\mu = -2$ there are 14 such vectors $\mathbf{b}$, namely the vectors $(1, 1, 1, 1, 0, 0, 0)^T$, $(1, 1, 0, 1, 1, 0, 0)^T$ and cyclic permutations thereof. On the other hand we know that C_7 is a star complement for -2 in the 21-vertex graph $L(K_7)$ and so we know without further calculation that (a) $\Gamma(C_7, -2) = K_{14}$, and (b) any graph with C_7 as a star complement for -2 is an induced subgraph of $L(K_7)$.

If $\mu = -1$ then $m(\mu) = -3$ and Equation (2) becomes $3(I + A) = B^T(J - 3P^2 - 3P^{-2})B$, where each entry of J is 1 and $P = \mathrm{circul}(-8, 1, 0, 0, 0, 0, 0)$. The two graphs which have an eigenvalue -1 in Table 2 account for 21 possible vectors $\mathbf{b}$ such that $\langle\langle \mathbf{b}, \mathbf{b} \rangle\rangle = 3$, namely the vectors $(1, 1, 1, 0, 0, 0, 0)^T$, $(1, 1, 0, 1, 0, 0, 0)^T$, $(1, 0, 1, 1, 0, 0, 0)^T$ and cyclic permutations thereof. Denote the corresponding sets of size 7 by X_1, X_2, X_3, respectively. (Note that we must consider two orientations of the 7-cycle in this case). We now compute $\langle\langle \mathbf{b}, \mathbf{c} \rangle\rangle$ for each pair $\mathbf{b}, \mathbf{c}$ of such vectors, and here we may exploit the fact that $\langle\langle P\mathbf{b}, P\mathbf{c} \rangle\rangle = \langle\langle \mathbf{b}, \mathbf{c} \rangle\rangle$. We find that $\langle\langle \mathbf{b}, \mathbf{c} \rangle\rangle \in \{0, 3\}$ for all but 28 pairs, representing precisely 28 non-edges in $\Gamma(C_7, -1)$. These account for a 7-cycle on X_2, a 7-cycle on X_3 and a 14-cycle with vertices alternately in X_2 and X_3. Recall that the maximal cliques in $\Gamma(C_7, -1)$ correspond to the maximal co-reduced graphs which have C_7 as a star complement for -1. We turn to the database to find that there are 35 such (labelled) graphs, corresponding to maximal cliques of size 11 or 12. Four non-isomorphic graphs arise: three have 18 vertices (each occurring 7 times) and one has 19 vertices (occurring 14 times).

We may proceed similarly with the remaining eigenvalue 0, excluding from consideration any graphs with isolated vertices. The extendability graph $\Gamma(C_7, 0)$

has 35 vertices, and we find from the database that there are 34 non-isomorphic maximal reduced graphs which arise: there are 25 non-isomorphic graphs with 20 vertices (21 occur 14 times, 4 occur 7 times); 5 non-isomorphic graphs with 21 vertices (each occurs 14 times); and 4 non-isomorphic graphs with 22 vertices (2 occur 14 times, 2 occur 7 times).

	λ_1	λ_2	λ_4	λ_6	λ_8
1	2.0912	1.4427	0.2163	–0.7764	–1.9738
12	2.4005	1.5864	0.0000	–1.1332	–1.8538
13	2.3387	1.4511	0.0000	–0.5191	–2.2706
14	2.3144	1.2871	0.5903	–1.0742	–2.1176
123	2.7480	1.5702	–0.3540	–1.0000	–1.9642
124	2.6611	1.4264	0.1194	–1.0000	–2.2068
125	2.6373	1.3241	0.4891	–1.5030	–1.9476
135	2.5991	1.2822	0.2031	–0.5929	–2.4915
1234	3.0472	1.4919	–0.3793	–1.1598	–2.0000
1235	2.9714	1.3732	0.0000	–1.0842	–2.2604
1245	2.9520	1.2968	0.3308	–1.5795	–2.0000
1246	2.9140	1.2700	0.0000	–0.6364	–2.5476
12345	3.3086	1.3815	–0.2210	–1.5367	–1.9324
12346	3.2582	1.3183	–0.3708	–0.7750	–2.4307
12356	3.2425	1.2575	0.0826	–1.3010	–2.2816
123456	3.5553	1.2819	–0.2975	–1.3434	–2.1963
1234567	3.8284	(1.2470)	(–0.4450)	(–1.8019)	–1.8284

Table 2

Example 3. For $H = C_6$ the only interesting eigenvalue is $\mu = 0$. The extendability graph has 29 vertices. There are two non-isomorphic maximal reduced graphs, one on 19 vertices and one on 20 vertices (note that isolated vertices are excluded). The first graph occurs 4 times since there are 4 maximal cliques of size 13 giving rise to this graph. The second graph appears 12 times. The spectra of these graphs are 7.2450, 2.8541^2, 0^{13}, -3.8541^2, 5.2450 and 7.9045, 3, 2.7988, 0^{14}, -3.5744, -4.5050, -5.6209.

Example 4. Let H be the star complement (9) from Example 1, i.e. $H = 2K_3 \cup K_1$, and consider the eigenvalues $\mu = 1$ and $\mu = -2$ which are relevant in this case.

For $\mu = 1$ the extendability graph has 16 vertices. Except for two uninteresting maximal graphs on 8 vertices, we obtain the graph G of Fig. 1, the original graph of Example 1. The graph G arises from 6 maximal cliques of size 3 in the extendability graph of H.

In the case $\mu = -2$ the extendability graph has 25 vertices. There are 512 maximal graphs on 17 vertices and just one on 22 vertices. Among those on 17 vertices there are 18 isomorphism classes: all are non-regular with a vertex adjacent to all other vertices. One of these graphs has only three distinct eigenvalues; its spectrum is $8,\ 2^6,\ -2^{10}$. The spectra of all other graphs are of the form $\alpha,\ 2^5,\ \beta,\ -2^{10}$ with $\alpha+\beta=10$. The maximal graphs which arise here include (non-isomorphic) cospectral graphs. The graph on 22 vertices has spectrum $13.1789,\ 3^5,\ 1.8211,\ -2^{15}$.

The final example is related to instances of Problem $2'$.

Example 5. If we take the twenty exceptional graphs on 6 vertices in the role of star complements for -2 we find in total ten non-isomorphic maximal graphs. Seven of them have 11 vertices, and for the remaining three the number of vertices is 12, 13 and 16 respectively. There is a non-regular graph on 11 vertices with 3 distinct eigenvalues; its spectrum is $5,\ 1^5,\ -2^5$. The remaining six graphs on 11 vertices have spectrum $\alpha,\ 1^4,\ \beta,\ -2^5$ with $\alpha+\beta=6$. Two of these graphs are cospectral. The graphs on 13 and 16 vertices are integral (i.e. have only integer eigenvalues); that with 16 vertices is the Clebsch graph which is strongly regular (with three distinct eigenvalues). Additional examples related to the exceptional graphs are described in [**11**].

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