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ON HUA'S INEQUALITY IN STRICTLY CONVEX NORMED SPACES

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A generalization of the celebrated Hua's inequality in strictly convex normed spaces and some related results are given.

1. INTRODUCTION

The following inequality due to LO-KENG HUA [**1**] is important in the number theory:

$$\left(\delta - \sum_{i=1}^{n} x_i\right)^2 + \alpha \cdot \sum_{i=1}^{n} {x_i}^2 \geq \frac{\alpha}{n+\alpha} \cdot \delta^2, \tag{1.1.}$$

where $\delta, \alpha > 0$, $x_i \in \mathbf{R}$ $(i = 1, 2, \ldots, n)$ with equality if and only if

$$x_i = \frac{1}{n+\alpha} \cdot \delta \qquad (i = 1, 2, \ldots, n).$$

Recently, S. S. DRAGOMIR and G-S. YANG [**2**] have given a generalization of (1.1) in real inner product spaces. Here we extend the result from [**2**] to strictly convex normed spaces.

2. MAIN RESULT

Theorem 2.1. *Let* $(X, \|.\|)$ *be a strictly convex normed space and let* $x_i \in X$ $(i = 1, 2, \ldots, n)$, $y \in X$ *and* $\alpha > 0$. *Then the inequality*

$$\left\| y - \sum_{i=1}^{n} x_i \right\|^2 + \alpha \cdot \sum_{i=1}^{n} \|x_i\|^2 \geq \frac{\alpha}{n+\alpha} \cdot \|y\|^2 \tag{2.1}$$

holds with equality if and only if $x_i = \dfrac{1}{n+\alpha} \cdot y$ *for* $i = 1, 2, \ldots, n$.

First we shall prove the following technical lemma, which is interesting in itself.

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Lemma 2.2. *Let* $(X, \|.\|)$ *be a strictly convex normed space, then*

$$\|y-x\|^2 + \alpha \cdot \|x\|^2 \geq \frac{\alpha}{1+\alpha}\|y\|^2, \tag{2.2}$$

where $\alpha > 0$, $x, y \in X$. *The equality in* (2.2) *holds if and only if* $x = \frac{1}{1+\alpha} \cdot y$.

Proof. We shall start with the inequality

$$(\delta - t)^2 + \alpha \cdot t^2 \geq \frac{\alpha}{1+\alpha} \cdot \delta^2, \tag{2.3}$$

where $t \in \mathbf{R}$, $\alpha, \delta > 0$, with equality if and only if $t = \delta/(1+\alpha)$. By triangle inequality and (2.3), we have

$$\|y-x\|^2 + \alpha \cdot \|x\|^2 \geq (\|y\| - \|x\|)^2 + \alpha\|x\|^2 \geq \frac{\alpha}{1+\alpha}\|y\|^2. \tag{2.4}$$

The equality in (2.4) holds if and only if

$$\|y-x\| = |\, \|y\| - \|x\| \,| \tag{2.5}$$

and

$$\|x\| = \frac{1}{1+\alpha} \cdot \|y\|. \tag{2.6}$$

Since $(X, \|.\|)$ is strictly convex normed spaces, from (2.5) we get $y - x = \lambda x$, $(y \neq x,\ \lambda > 0)$, i.e.

$$x = \frac{1}{1+\lambda} \cdot y \tag{2.7}$$

Putting (2.7) in (2.6), we get $\lambda = \alpha$, i.e. $x = y/(1+\alpha)$. □

Proof of the theorem. We shall use the induction argument. For $n = 1$ it is true by Lemma 2.2. Applying Lemma 2.2. we next have

$$\begin{aligned} \left\| y - \sum_{i=1}^{n+1} x_i \right\|^2 + \alpha \cdot \sum_{i=1}^{n+1} \|x_i\|^2 &= \left\| \left(y - \sum_{i=1}^{n} x_i \right) - x_{n+1} \right\|^2 + \alpha \cdot \|x_{n+1}\|^2 + \alpha \cdot \sum_{i=1}^{n} \|x_i\|^2 \\ &\geq \frac{\alpha}{1+\alpha} \left\| y - \sum_{i=1}^{n} x_i \right\|^2 + \alpha \cdot \sum_{i=1}^{n} \|x_i\|^2 \\ &= \frac{\alpha}{1+\alpha} \cdot \left(\left\| y - \sum_{i=1}^{n} x_i \right\|^2 + (1+\alpha) \cdot \sum_{i=1}^{n} \|x_i\|^2 \right) \\ &\geq \frac{\alpha}{1+\alpha} \cdot \frac{1+\alpha}{n+1+\alpha} \cdot \|y\|^2 = \frac{\alpha}{n+1+\alpha} \cdot \|y\|^2. \end{aligned} \tag{2.8}$$

The equality in the (2.8) holds if and only if

$$x_{n+1} = \frac{1}{1+\alpha}\left(y - \sum_{i=1}^{n} x_i\right) \quad \text{and} \quad x_i = \frac{1}{n+1+\alpha} \cdot y \quad (i = 1, 2, \ldots, n)$$

i.e. $x_i = \dfrac{1}{n+1+\alpha} \cdot y \ \ (i = 1, 2, \ldots, n, n+1)$. $\square$

Remark 2.3. *If $(X, \|.\|)$ is not a strictly convex space then the equality in (2.1) does not imply $x_i = \dfrac{1}{n+\alpha} \cdot y$ $(i = 1, 2, \ldots, n)$. For example, let c_0 be the space of all complex sequences that converge to zero. Then $(c_0, \|.\|)$ is not strictly convex. To see this let $y = (\eta_\nu)_{\nu=1,2,\ldots}$ and $x_i = (\xi_\nu^{\,i})_{\nu=1,2,\ldots}$, where*

$$\eta_\nu = \begin{cases} 1 & : \ \nu = 1 \\ \dfrac{n-\nu+2}{n+\alpha} & : \ \nu = 2, 3, \ldots, n+1 \\ 0 & : \ \nu = n+2, n+3, \ldots \end{cases}$$

and

$$\xi_\nu^{\,i} = \begin{cases} \dfrac{1}{n+\alpha} & : \ \nu = 1, 2, \ldots, i+1 \\ \\ 0 & : \ \nu = i+1, i+3, \ldots \end{cases} \qquad (i = 1, 2, \ldots, n).$$

Then $y, x_i \in c_0$ and the equality in (2.1) is true whereas

$$x_i \neq \frac{1}{n+\alpha} \cdot y \quad (i = 1, 2, \ldots, n).$$

Remark 2.4. *We see that Lemma 2.2 gives the criterion for some normed spaces to be strictly convex.*

From Theorem 2.1 we easily get

Corollary 2.5. *Let $(X, \|.\|)$ be a strictly convex normed space and let*

$$\left\| y - \frac{1}{n} \cdot \sum_{i=1}^{n} x_i \right\| = O\left(\frac{1}{n^\lambda}\right) \qquad \left(\lambda > \frac{1}{2}\right)$$

hold. Then $\underline{\lim}_{n\to\infty}\, n \cdot \sum_{i=1}^{n} \|x_i\|^2 \geq \|y\|^2$, where $y, x_i \in X$ $(i = 1, 2, \ldots, n)$.

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