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LALESCU SEQUENCES

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The convergence of some sequences related to the Lalescu sequences is studied.

The Romanian mathematical journal Gazeta Mathematică (Bukarest) appears monthly since 1895. In one of the first volumes, more exactly in a number from 1900 (see [**6**]), T.LALESCU has proposed, as problem 579, the study of a sequence with the general term

$$L_n = \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}.$$

It is called now LALESCU sequence, at least by Romanian mathematicians, and many variants of it have appeared during this century in the same journal. The first one was considered as the problem 2042 (see [**5**]) and has the general term

$$I_n = (n+1)\sqrt[n+1]{(n+1)} - n\sqrt[n]{n}.$$

We relate to them also a third sequence given by

$$J_n = \frac{(n+1)^n}{n^{n-1}} - \frac{n^{n-1}}{(n-1)^{n-2}}$$

which appears as the problem 4600 (see [**7**]). At the end of this paper we will give some other sequences which have appeared in the last years.

We begin by indicating a general result giving the limit of a sequence which looks like the sequences mentioned above. The method of proof is that used in the first published solution for LALESCU's problem. This was forget and many other more sophisticated solutions were considered (see [**1**] for more information).

We study sequences with the general term

$$x_n = y_n - z_n$$

where

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \infty, \qquad \lim_{n\to\infty} \frac{y_n}{z_n} = 1.$$

Theorem 1. *If there exist the positive constants b and c such that*

$$\lim_{n\to\infty} \frac{z_n}{n^c} = z > 0, \qquad \lim_{n\to\infty} \left(\frac{y_n}{z_n}\right)^{n^b} = y > 0,$$

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then

$$\lim_{n\to\infty}(y_n - z_n) = \begin{cases} 0 & (c < b), \\ z\ln y & (c = b), \\ \infty & (c > b, y > 1), \\ -\infty & (c > b, y < 1). \end{cases}$$

Proof. We write

$$y_n - z_n = \frac{\dfrac{y_n}{z_n} - 1}{\ln\left(\dfrac{y_n}{z_n}\right)} \frac{z_n}{n^c} n^{c-b} \ln\left(\frac{y_n}{z_n}\right)^{n^b}$$

and use the hypotheses and the well known result $\lim_{n\to\infty}\dfrac{t-1}{\ln t} = 1$

Example 1. The sequence

$$x_n = \frac{(n+p)^n}{n^{n-k}} - \frac{n^{n-h}}{(n-p)^{n-h-k}}$$

has a finite limit if and only if $k = 1$ and in this case the limit is $pe^p(h + k - p)$. Indeed, in this case

$$y_n = \frac{(n+p)^n}{n^{n-k}}, \qquad z_n = \frac{n^{n-h}}{(n-p)^{n-h-k}}$$

and so

$$\lim_{n\to\infty}\frac{y_n}{n^k} = e^p$$

while

$$\lim_{n\to\infty}\left(\frac{y_n}{z_n}\right)^n = \lim_{n\to\infty}\left(\left(\frac{n+p}{n}\right)^{h+k}\left(\frac{n^2-p^2}{n^2}\right)^{n-h-k}\right)^n q = e^{p(h+k-p)}.$$

Taking $p = k = h = 1$ we get the sequence (J_n) with limit e.

To study the sequences (L_n) or (I_n) we want to apply Theorem 1 to a sequence with the general term

$$x_n = \sqrt[n+1]{p_{n+1}} - \sqrt[n]{q_n}.$$

Theorem 2. *If the positive sequences* (p_n) *and* (q_n) *have the infinite limits and for some* $c > 0$ *satisfy*

$$\lim_{n\to\infty}\frac{p_{n+1}}{n^c p_n} = p > 0, \qquad \frac{q_n}{p_n} = q > 0$$

then

$$\lim_{n\to\infty}\left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{q_n}\right) = \begin{cases} 0 & (c < 1), \\ \dfrac{p}{e}\ln\dfrac{e}{q} & (c = 1), \\ \infty & (c > 1, q < e^c), \\ -\infty & (c > 1, q > e^c). \end{cases}$$

Proof. We take $y_n = \sqrt[n+1]{p_{n+1}}, \quad z_n = \sqrt[n]{q_n}$. Of course

$$\lim_{n\to\infty} \frac{q_{n+1}}{n^c q_n} = \lim_{n\to\infty} \frac{q_{n+1}}{p_{n+1}} \frac{p_{n+1}}{n^c p_n} \frac{p_n}{q_n} = p$$

and so

$$\begin{aligned} \lim_{n\to\infty} \frac{z_n}{n^c} &= \lim_{n\to\infty} \sqrt[n]{\frac{q_n}{n^{nc}}} = \lim_{n\to\infty} \frac{q_{n+1}}{(n+1)^{(n+1)c}} \frac{n^{nc}}{q_n} \\ &= \lim_{n\to\infty} \frac{q_{n+1}}{n^c q_n} \left(\frac{n}{n+1}\right)^{(n+1)c} = \frac{p}{e^c}. \end{aligned}$$

We have used the well known implication

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = 1.$$

Also

$$\lim_{n\to\infty} \left(\frac{y_n}{z_n}\right)^n = \lim_{n\to\infty} \frac{p_{n+1}}{n^c p_n} \frac{n^c}{\sqrt[n+1]{p_{n+1}}} \frac{p_n}{q_n} = \frac{e^c}{q}.$$

thus we can apply Theorem 1 with $b = 1$.

We use in what follows only a special case of this result.

Consequence. *If the positive sequence* (p_n) *is such that*

$$\lim_{n\to\infty} \frac{p_{n+1}}{np_n} = p > 0$$

then

$$\lim_{n\to\infty} \left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{p_n} \right) = \frac{p}{e}.$$

With its help we can find the limit of some sequences given in the above mentioned journal Gazeta Matematica. First of all, for $p_n = n!$ we get the sequence (L_n) with limit $1/e$ and for $p_n = n^{n+1}$ we get (I_n) with limit 1. If $p_n = n^{2n}/n!$ we have a sequence given in [**2**] with limit e. Taking $p_n = \Gamma\left(\frac{n+1}{2}\right)$ we get the sequence from [**4**] having limit $1/e$. Also for $p_n = \sqrt[3]{n!n^{2n}}$ we have the sequence given in [**3**] with limit $1/\sqrt[3]{e}$ and the list of examples can be continued.

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