Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. 9 (1998), 19–23.

ON POSSIBLE COMMUTING GENERALIZED INVERSES OF MATRICES

Jovan D. Kečkić

Let M be the multiplicative semigroup of all complex square matrices of a fixed order. In this note we show that if $A \in M$, then the only possibile generalized inverse of A which commutes with A is the Drazin inverse A^D.

1. Let M be the set of all complex square matrices of a fixed order. For any $A \in M$ and any $k \in \mathbf{N}$ the system of equations in X:

$$A^{k+1}X = A^k, \quad AX = XA, \quad AX^2 = X \tag{1.1}$$

can have at most one solution.

The index of a matrix A, $\operatorname{Ind} A$, is defined as the smallest positive integer such that $\operatorname{rank} A^{\operatorname{Ind} A} = \operatorname{rank} A^{1+\operatorname{Ind} A}$.

The system (1.1) is consistent if and only if $\operatorname{Ind} A \le k$. Its unique solution is called the Drazin inverse of A and is denoted by A^D.

If A is nilpotent, than $A^D = 0$ and if A is regular than $A^D = A^{-1}$. If A is neither nilpotent nor regular there exist regular matrices S, R and a nilpotent matrix N such that $A = S(N \oplus R)S^{-1}$. Then $A^D = S(0 \oplus R^{-1})S^{-1}$.

All this is well known; see, for instance [**1**].

If we treat M as the multiplicative semigroup, than any term made up from A and X has the form

$$A^{m_1}X^{n_1}A^{m_2}X^{n_2}\cdots A^{m_p}X^{n_p} \quad (m_i, n_i \in \mathbf{N}_0). \tag{1.2}$$

We say that the equation $A^{m_1}X^{n_1}A^{m_2}X^{n_2}\cdots A^{m_p}X^{n_p} = A^{m'_1}X^{n'_1}A^{m'_2}X^{n'_2}\cdots A^{m'_q}X^{n'_q}$ is balanced if

$$m_1+m_2+\cdots+m_p - (n_1+n_2+\cdots+n_p) = m'_1+m'_2+\cdots+m'_q - (n'_1+n'_2+\cdots+n'_q).$$

A balanced equation becomes an identity if A regular and if $X = A^{-1}$.

1991 Mathematics Subject Classification: 15A09

We say that system of equations in X:

$$t_1(A,X)=t_1'(A,X),\dots,t_r(A,X)=t_r'(A,X), \tag{1.3}$$

where t_i, t_i' are terms of the form (1.2) is balanced if each one of the equations which appear in (1.3) is balanced.

The system (1.1) is balanced and for any $A\in M$ it cannot have more than one solution. Furthermore, it is consistent not only for regular but also for some singular matrices. Hence, it defines a generalized inverse of A.

If system (1.3) is balanced, if for any $A\in M$ it cannot have more than one solution, and if it is consistent for at least one singular matrix A, we say that it defines a generalized inverse of A.

In this note we investigate those systems (1.3) which define a generalized inverse of A and which contain the equation $AX=XA$; in other words we look for all possible commuting generalized inverses of A (in the multiplicative semigroup M).

2. If $AX=XA$, then any multiplicative term made up from A and X has the form A^mX^n, and a balanced equation must be of the form $A^{m+p}X^{n+p}=A^mX^n$. This means that if a system is to define a commuting generalized inverse, it must have the form

$$AX=XA,\quad A^{m_1+p_1}X^{n_1+p_1}=A^{m_1}X^{n_1},\ \dots,\ A^{m_r+p_r}X^{n_r+p_r}=A^{m_r}X^{n_r}, \tag{2.1}$$

where $m_i,n_i\in\mathbf{N}_0,\ p_i\in\mathbf{N}$.

If $m_i>0$ or $n_i>1$ for all $i\in\{1,\dots,r\}$ the system (2.1) can have more than one solution. Indeed, if all $m_i>0$, than for $A=0$ arbitrary $X\in M$ is a solution of (2.1). If all $n_i>1$, than for $A=0$ all matrices X such that $X^n=0$ where $n=\min n_i$, satisfy (2.1).

We therefore suppose that there exists $i\in\{1,\dots,r\}$ such that $m_i=0$ and $n_i\le 1$, i.e. such that $m_i=n_i=0$ or $m_i=0$, $n_i=1$. Of course, we may take $i=1$.

If $m_1=n_1=0$ the system (2.1) becomes

$$AX=XA,\quad A^{p_1}X^{p_1}=I,\quad A^{m_i+p_i}X^{n_i+p_i}=A^{m_i}X^{n_i}\quad (i=2,\dots,r). \tag{2.2}$$

The equation $A^{p_1}X^{p_1}=I$ implies that the system (2.2) is inconsistent if A is singular. Hence this system does not define a generalized inverse of A.

We now consider the system

$$AX=XA,\quad A^pX^{p+1}=X,\quad A^{m_i+p_i}X^{n_i+p_i}=A^{m_i}X^{n_i}\qquad (i=2,\dots,r). \tag{2.3}$$

obtained from (2.1) for $m_1=0$, $n_1=1$, $p_1=p$.

We distinguish between three cases. Let A be nilpotent with $A^{n-1} \neq 0$, $A^n = 0$. If $n \leq p$ the unique solution of $A^p X^{p+1} = X$ is given by $X = 0$. If $n > p$, there exists a positive integer q such that $qp < n \leq (q+1)p$. We then have

$$\begin{aligned} A^p X^{p+1} = X &\Rightarrow A^{n-p} A^p X^{p+1} = A^{n-p} X = 0, \\ A^p X^{p+1} = X &\Rightarrow A^{n-2p} A^p X^{p+1} = A^{n-2p} X = 0, \\ &\vdots \\ A^p X^{p+1} = X &\Rightarrow A^{n-qp} A^p X^{p+1} = A^{n-qp} X = 0, \end{aligned}$$

and so we again get $X = A^p X^{p+1} = A^{(q+1)p-n}(A^{n-qp}X)X^p = 0$. Hence, if (2.3) is consistent, it has unique solution: $X = 0$.

If A is regular, the sistem (2.3) becomes

$$(2.4) \qquad AX = XA, \quad A^p X^{p+1} = X, \quad A^{p_i} X^{n_i+p_i} = X^{n_i} \qquad (i = 2, \ldots, r)$$

and unless $n_i = 0$ for some $i \in \{2, \ldots, r\}$, it has at least two solutions: $X = 0$ and $X = A^{-1}$.

Suppose that A is neither nilpotent nor regular. Then there exist regular matrices S, R and a nilpotent matrix N such that $A = S(N \oplus R)S^{-1}$. Let

$$(2.5) \qquad X = S \begin{Vmatrix} P & U \\ V & Q \end{Vmatrix} S^{-1}$$

where P and N, and Q and R are of the same order.

From the equation $AX = XA$ we get $NP = PN$, $NU = UR$, $RV = VN$, $RQ = QR$. However,

$$NU = UR \Rightarrow U = NUR^{-1} = N(NUR^{-1})R^{-1} = N^2UR^{-2} = N^3UR^{-3} = \ldots = 0,$$

since N is nilpotent, and similarly we get $V = 0$. Hence, $X = S(P \oplus Q)S^{-1}$ and $NP = PN$, $RQ = QR$.

From the equation $A^p X^{p+1} = X$ we get $N^p P^{p+1} = P$ and $R^p Q^{p+1} = Q$. The first of those two equations implies

$$P = N^p P^{p+1} = N^p P P^p = N^p (N^p P^{p+1}) P = N^{2p} P^{2p+1} = N^{3p} P^{3p+1} = \ldots = 0,$$

since N is nilpotent.

Hence, (2.5) becomes $X = S(0 \oplus Q)S^{-1}$ and for the matrix Q from (2.3) we obtain the following system of equations

$$R^p Q^{p+1} = Q, \quad R^{p_i} Q^{n_i+p_i} = Q^{n_i} \qquad (i = 2, \ldots, r)$$

and unless $n_i = 0$ for some $i \in \{2, \ldots, r\}$ it has at least two solutions: $Q = 0$ and $Q = R^{-1}$.

3. Therefore, let $m_2 \neq 0$, $n_2 = 0$. The system (2.3) becomes

$$(3.1) \quad AX = XA, \quad A^p X^{p+1} = X, \quad A^{m_2+p_2} X^{p_2} = A^{m_2}, \quad A^{m_i+p_i} X^{n_i+p_i} = A^{m_i} X^{n_i}$$

where $i = 3, \ldots, r$.

If A is nilpotent, we know that $A^p X^{p+1} = X$ implies $X = 0$, and hence the system (3.1), if consistent, has unique solution.

If A is regular, the third equation of (3.1) reduces to $A^{p_2} X^{p_2} = I$, which means that X is also regular and (3.1) becomes

$$(3.2) \qquad AX = XA, \quad A^p X^p = I, \quad A^{p_i} X^{p_i} = I \qquad (i = 2, \dots, r)$$

Denote by $(p_1, p_2, \dots, p_r)$ the highest common factor of $p_1, \dots, p_r$. The system (3.2) has unique solution if and only if $(p, p_2, \dots, p_r) = 1$. Indeed, if $(p_\mu, p_\nu) = 1$ there exist positive integers u and v such that $up_\mu - vp_\nu = 1$. Hence,

$$AX = (AX)^{p_\mu u - p_\nu v} = ((AX)^{p_\mu})^u ((AX)^{p_\nu})^{-v} = I,$$

and so $X = A^{-1}$. If $(p, p_2, \dots, p_r) > 1$ the system (3.2) can have more than one solution.

If A is neither nilpotent nor regular, let $A = S(N \oplus R)S^{-1}$, where S, N, R are as before. Then, as we know $X = S(0 \oplus Q)S^{-1}$, where

$$(3.3) \quad RQ = QR, \ \ R^p Q^{p+1} = Q, \ \ R^{p_2} Q^{p_2} = I, \ \ R^{p_i} Q^{n_i + p_i} = Q^{n_i} \ \ (i = 3, \dots, r).$$

However, the equality $R^{p_2} Q^{p_2} = I$ implies that Q is regular and the system (3.3) reduces to

$$RQ = QR, \quad R^p Q^p = I, \quad R^{p_i} Q^{p_i} = I \qquad (i = 2, \dots, r)$$

and it has unique solution $Q = R^{-1}$ provided, as before, that $(p, p_2, \dots, p_r) = 1$.

We have therefore proved the following

Theorem 1. *The system* (2.1) *defines a generalized inverse of* A *if and only if:*

(i) there exist $i, j \in \{1, \dots, r\}$, $i \neq j$, *such that* $m_i = 0$, $n_i = 1$, $m_j \neq 0$, $n_j = 0$;

(ii) $(p_1, \dots, p_r) = 1$.

4. Consider now the system

$$(4.1) \qquad \begin{cases} AX = XA, \quad A^p X^{p+1} = X, \quad A^{m_i + p_i} X^{p_i} = A^{m_i} \quad (i = 1, \dots, t) \\ A^{m_i + p_i} X^{n_i + p_i} = A^{m_i} X^{n_i} \qquad (i = t+1, \dots, r) \end{cases}$$

where $m_1, \dots, m_t \geq 1$, $n_{t+1}, \dots, n_r \geq 1$ $p, p_1, \dots, p_r \geq 1$, and let

$$(4.2) \qquad \operatorname{Ind} A \leq \min(m_1, \dots, m_t).$$

If A is nilpotent and if (4.2) holds, then $A^{m_i} = 0$ for all $i = 1, \dots, t$ and a solution of (4.1) is given by $X = 0$.

If A is regular, then Ind $A = 0$, and (4.2) is true. A solution of (4.1) is given by $X = A^{-1}$.

If $A = S(N \oplus R)S^{-1}$, where N is nilpotent and S, R are regular the first two equations of (4.1) imply $X = S(0 \oplus Q)S^{-1}$. From the remaining equations we get

$$N^{m_i} = 0 \qquad (i = 1, \dots, t) \tag{4.3}$$

and

$$R^{p_i} Q^{p_i} = I \qquad (i = 1, \dots, r)$$

and (4.1) has a solution, e.g. $X = S(0 \oplus R^{-1})S^{-1}$, if and only if the equalities (4.3) hold.

Therefore, we have

Theorem 2. *The system* (4.1) *is consistent if and only if* (4.2) *is true.*

We have seen that if the system (4.1) is consistent and if it has unique solution, this solution is given by $X = A^D$. Hence, we have

Theorem 3. *If* $(p_1, \dots, p_r) = 1$ *the system* (4.1) *is equivalent to the system* (1.1) *where* $k = min\,(m_1, \dots, m_t)$.

From Theorems 1, 2 and 3 we conclude that in the multiplicative semigroup M the Drazin inverse A^D is the only possible generalized inverse which commutes with A.

REFERENCES

1. S. L. Campbell, C. D. Meyer, Jr.: *Generalized Inverses of Linear Transformations*, Pitman, London - San Francisco - Melbourne, 1979.

Tikveška 2,
11000 Belgrade,
Yugoslavia

(Received December 5, 1996)