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EXTENSIONS OF AN INEQUALITY

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We discuss several extensions of inequality (1).

The inequality

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \tag{1}$$

holds for sides of a triangle since it can be rewritten in the form

$$(b+c-a)(c-a)^2 \ + \ (c+a-b)(a-b)^2 \ + \ (a+b-c)(b-c)^2 \geq 0. \tag{1$'$}$$

Since the inequality cannot be extended to arbitrary non-negative numbers a, b, c, I had proposed the problem (♯2064) [**1**] of proving that a valid extension is

$$3\max\left\{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right), \left(\frac{b}{a}+\frac{a}{c}+\frac{c}{b}\right)\right\} \geq (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right). \tag{2}$$

Actually, (2) can be strengthened to

$$3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{b}{a}+\frac{a}{c}+\frac{c}{b}\right) \geq 2(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \tag{3}$$

since it is equivalent to

$$\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right) \geq 6 \tag{3$'$}$$

and which was noted by a number of solvers of the problem as well as muself.

One of the solvers CHRISTOPHER BRADLEY proved that (1) even holds whenever $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ are sides of a triangle by employing equivalently the known triangle inequality

$$x^2+y^2+z^2 \geq 2yz\cos A + 2zx\cos B + 2xy\cos C, \tag{4}$$

where x, y, z are arbitrary real numbers and A, B, C are angles of a triangle. When I had submitted the problem of inequality (2), I had forgotten the BRADLEY result which is equivalent to

$$3\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right) \geq (a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \tag{5}$$

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for a, b, c being sides of a triangle was noted in a paper of mine [**2**] and had been proposed previously by A. W. Walker [**3**]. My proof was by showing (5) was equivalent to

$$(5') \qquad 3(A\Omega^2 + B\Omega^2 + C\Omega^2) \geq (a^2 + b^2 + c^2)$$

where Ω is a Brocard point of triangle ABC. Actually the latter is valid for any point Ω in the plane or out of the plane of ABC and follows from the obvious inequality

$$(x\boldsymbol{A} + y\boldsymbol{B} + z\boldsymbol{C})^2 \geq 0,$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are vectors from an arbitrary point P to the vertices A, B, C, respectively, and x, y, z are arbitrary real numbers. Expanding out, the latter inequality reduces to the known polar moment of inertia inequality

$$(6) \qquad (x + y + z)(xPA^2 + yPB^2 + zPC^2) \;\geq\; yza^2 + zxb^2 + xyc^2$$

(now just set $x = y = z$). There is equality in $(5')$ (excluding degenerate triangles) if and only if Ω coincides with the centroid and this only occurs if the triangle is equilateral.

We now extend (2) by determining all triples a, b, c of positive numbers such that

$$(7) \qquad 3\min\left\{\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right), \left(\frac{b}{a} + \frac{a}{c} + \frac{c}{b}\right)\right\} \geq (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Since the inequality is homogeneous, we can assume without loss of generality that $a \geq b \geq c = 1$ $\left(\text{here } \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leq \left(\frac{b}{a} + \frac{a}{c} + \frac{c}{b}\right)\right)$. Now letting, $b = 1 + s,\ a = 1 + s + r$ where $r, s \geq 0$, (7) reduces after some algebra to

$$(7') \qquad r^2(1 - s) + rs + s^2 + s^3 \geq 0.$$

Clearly if $s \leq 1$, the latter holds for all r. For $s > 1$, $(7')$ will be valid only if

$$(8) \qquad r \;\leq\; \frac{s}{2(s-1)}\left(1 + \sqrt{4s^2 - 3}\right)$$

which is gotten by solving the quadratic in r. As an example, setting $s = 2,\ \ r \leq 1 + \sqrt{13}$, so if we set $c = 1$, $b = 3$, and $a = 4 + \sqrt{13}$, we get equality in (7).

Using (8) we can obtain another proof that (7) is valid if $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are sides of a triangle. The extreme case here is if the sides are 1, t^2 and $(t + 1)^2$, so that $1 + s = t^2 (> 2)$ and $1 + s + r = (t + 1)^2$. Since this requires that $r = \left(1 + \sqrt{s + 1}\right)^2 - s - 1$, it suffices to show that

$$\left(1 + \sqrt{s+1}\right)^2 - s - 1 \leq \frac{s}{2(s-1)}\left(1 + \sqrt{4s^2 - 3}\right).$$

On replacing s by $t^2 - 1$ and squaring we obtain

$$(t^2 - 2)(t^3 - 3t - 1)^2 \geq 0$$

and there is equality if t is the positive root t_r of the cubic and is ~ 1.879389017.

In the solution of problem 2064 [**1**], BILL SANDS (University of Calgary) had raised the following two interesting open problems:

(9) Find the largest t so that

$$(1-t)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+t\left(\frac{a}{b}+\frac{b}{b}+\frac{c}{a}\right)\geq\frac{1}{3}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

(here as before $a \geq b \geq c > 0$);

(10) find the smallest t so that (7) holds whenever a^t, b^t, c^t, are the sides of a triangle (or equivalently the largest n such that

$$3\left(\frac{a^n}{b^n}+\frac{b^n}{c^n}+\frac{c^n}{a^n}\right)\geq\frac{a^n+b^n+c^n}{\dfrac{1}{a^n}+\dfrac{1}{b^n}+\dfrac{1}{c^n}},$$

where a, b, c are sides of a triangle).

For (9), we show that t must be at most $2/3$ and to prove this we proceed as before by letting $c = 1, b = 1+s, a = 1+r+s$ so that (9) becomes

$$2r^2s+3rs^2+s^3+s^2+rs+r^2\geq 3t\left(rs^2+r^2s\right).$$

Since by a proper choice of r and s, r^2s can be the dominant term, t is at most $2/3$.

For (10), we show the smallest t is $1/2$ or equivalently the largest n is 2. Before we showed that if $1+s = t_r^2$, then

$$(11)\qquad \left(1+\sqrt{s+1}\right)^2-s-1=\frac{s}{2(s-1)}\left(1+\sqrt{4s^2-3}\right).$$

Now let $c = 1$, $b = x^n = 1+s = t_r^2$, $a = (x+1)^n = 1+s+r'$ for $n > 2$, so that

$$r'=\left(1+(1+s)^{1/n}\right)^n-s-1$$

and which by a previous analysis must be $\leq$ the right hand side of (11). But since r' is an increasing function of n for $s > 1$ (just consider the derivative with respect to n), it will be greater than the left hand side of (11).

Left open is more general problem of determining conditions on $\{x_i\}$ with $x_1 \geq x_2 \geq \ldots \geq x_n > 0$ such that

$$(12)\qquad n\left(\frac{x_1}{x_2}+\frac{x_2}{x_3}+\cdots+\frac{x_n}{x_1}\right)\geq(x_1+x_2+\cdots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right).$$

Note that requiring here that the x_i's are sides of a polygon is not sufficint. As an example, for $n = 4$, take $(x_1, x_2, x_3, x_4) = (4, 3, 2, 1)$. However, $(\sqrt{4}, \sqrt{3}, \sqrt{2}, 1)$ and $(\sqrt{3}, 1, 1, 1)$ are valid sets and these are such that the squares are sides of a quadrilateral.

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