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A QUALITATIVE STUDY ABOUT LOBACHEVSKY'S FUNCTIONAL EQUATION OF VECTORIAL ARGUMENT

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The purpose of this paper is to give some properties of the solutions of Lobachevsky's functional equation in the case: $f : E^n \to \mathbf{R}$, E^n-Euclidean n-dimensional real space and to establish the connections of this equation with some other functional equations in the same case.

1. Let E^n be a Euclidean real n-dimensional space in which we have

$$x = (\xi^1, \xi^2, \ldots, \xi^n),\ y = (\eta^1, \eta^2, \ldots, \eta^n);\ \xi^k,\ \eta^k \in \mathbf{R},\ k = 1, \ldots, n;$$

x, y are two vectors of E^n

$$0_{E^n} = (0, 0, \ldots, 0), \quad x = y \Leftrightarrow \xi^k = \eta^k \text{ for } k = 1, \ldots, n,$$
$$x + y = (\xi^1 + \eta^1, \xi^2 + \eta^2, \ldots, \xi^n + \eta^n),\ \lambda x = (\lambda\xi^1, \lambda\xi^2, \ldots, \lambda\xi^n)$$
$$\text{for all } x, y \in E^n, \lambda \in \mathbf{R},$$
$$\langle x, y\rangle = \sum_{k=1}^{n} \xi^k \eta^k \text{ is the scalar product of vectors } x, y \in E^n,$$
$$d(x, y) = \sqrt{\sum_{k=1}^{n} (\xi^k - \eta^k)^2} \text{ is the Euclidean distance between } x, y \in E^n,$$
$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\sum_{k=1}^{n} (\xi^k)^2} \text{ is the Euclidean norm of vector } x \in E^n.$$

$B^C_{E^n} = [e_1, e_2, \ldots, e_n]$ is the orthonormal basis of E^n, where $e_1 = (1, 0, 0, \ldots, 0, 0)$, $e_2 = (0, 1, 0, \ldots, 0, 0)$, $\ldots$, $e_n = (0, 0, 0, \ldots, 0, 1)$ are unit vectors;

$|\langle x, y\rangle| \le \|x\|\,\|y\|$ is CAUCHY–SCHWARZ–BUNIAKOWSKY inequality,

$$B(a; r) = \{x \in E^n \,|\, d(x, a) < r\} \text{ is open sphere (globe).}$$

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2. Let f be a function

$$f : E^n \to \mathbf{R},\ x = (\xi^1, \xi^2, \dots, \xi^n) \to f(x) = f(\xi^1, \xi^2, \dots, \xi^n).$$

The functional equation

$$(1) \qquad f(\xi^1, \xi^2, \dots, \xi^n) \cdot f(\eta^1, \eta^2, \dots, \eta^n) = f\left(\frac{\xi^1 + \eta^1}{2}, \frac{\xi^2 + \eta^2}{2}, \dots, \frac{\xi^n + \eta^n}{2}\right)^2$$

or shortly,

$$(1') \qquad f(x) \cdot f(y) = f\left(\frac{x+y}{2}\right)^2,$$

for all $x, y \in E^n$, is an extension of LOBACHEVSKY's functional equation [**1**] in the case $x, y \in \mathbf{R}$. We highlight some properties of the solution f of functional equation (1) which are analogous to the one dimensional case [**3**].

Lemma 1. *Let f be a solution of* (1)*. If there exists $x_0 \in E^n$ so that $f(x_0) = 0$, then $f(x) = 0$, for all $x \in E^n$ and if $f(0_{E^n}) \neq 0$, then*

$$(2) \qquad \operatorname{sgn} f(x) = \operatorname{sgn} f(0_{E^n}),$$

for all $x \in E^n$.

Proof. From (1) we obtain $f(x_0)f(2x - x_0) = f(x)^2$; i.e. $f(x) = 0$, for all $x \in E^n$. If $f(0_{E^n}) \neq 0$, then $f(0_{E^n})f(x) = f(x/2)^2 > 0$ which implies (2).

Lemma 2. *Let f, $f(0_{E^n}) \neq 0$ be a solution of* (1)*. The function f is continuous in E^n if and only if f is continuous in 0_{E^n}.*

Proof. The implication $\Rightarrow$ is obvious. The implication $\Leftarrow$ results in the following way: from (1) and Lemma 1 we have

$$(3) \qquad f(x) - f(x_0) = \frac{f\left(\frac{x - x_0}{2}\right)^2 - f(0)^2}{f(-x_0)}.$$

Because f is continuous in 0_{E^n} it results that f^2 is continuous in 0_{E^n} and from (3) we obtain that f is continuous for all $x \in E^n$.

Lemma 3. *Let f, $f(0_{E^n}) \neq 0$ be a solution of* (1)*. If f is bounded in $B(0_{E^n}; r) \subset E^n$, then f is continuous in E^n.*

Proof. *We consider the case $f(0_{E^n}) > 0$.* From (1) we successively obtain

$$f\left(\frac{x}{2}\right) = f(x)^{1/2} f(0_{E^n})^{1/2}, \quad f\left(\frac{x}{2^2}\right) = f\left(\frac{x}{2}\right)^{1/2} f(0_{E^n})^{1/2} = f(x)^{1/2^2} f(0_{E^n})^{1-1/2^2}$$

and by induction

$$f\left(\frac{x}{2^n}\right) = f(x)^{1/2^n} f(0_{E^n})^{1-1/2^n}. \tag{4}$$

Because f is bounded in $B(0_{E^n}; r)$, we have for all x

$$x \in B(0_{E^n}; r) \Leftrightarrow ||x|| = \sqrt{\sum_{k=1}^{n} (\xi^k)^2} \le r \Rightarrow f(x) \le M f(0_{E^n})$$

and $\lim\limits_{n\to\infty} \dfrac{x}{2^n} = 0_{E^n}$ for all $x \in B(0_{E^n}; r)$. Indeed, $d\left(\dfrac{x}{2^n}, 0_{E^n}\right) < \eta \Leftrightarrow \dfrac{||x||}{2^n} < \dfrac{r}{2^n} < \eta$ or $2^n > \dfrac{r}{\eta}$, i.e. exists $N(\eta) = \log^2 \dfrac{r}{\eta} + 1$ so that for $n > N(\eta) \Rightarrow d\left(\dfrac{x}{2^n}, 0_{E^n}\right) < \eta$, for all $x \in B(0_{E^n}; r)$ with $r/2^n < \eta$. On the other side, we have

$$\begin{aligned} d\left(f\left(\frac{x}{2^n}\right), f(0_{E^n})\right) &= \left|f\left(\frac{x}{2^n}\right) - f(0_{E^n})\right| \le \left|f(x)^{1/2^n} f(0_{E^n})^{1-1/2^n} - f(0_{E^n})\right| \\ &= f(0_{E^n}) \left|\left(\frac{f(x)}{f(0_{E^n})}\right)^{1/2^n} - 1\right| \le f(0_{E^n}) \left|M^{1/2^n} - 1\right|. \end{aligned}$$

Because $\lim\limits_{n\to\infty} M^{1/2^n} = 1$ for all $\varepsilon > 0$, exits $N(\varepsilon)$ so that for $d\left(\dfrac{x}{2^n}, 0_{E^n}\right) < \eta$, $n > N(\varepsilon)$ we have $|M^{1/2^n} - 1| < \dfrac{\varepsilon}{f(0_{E^n})}$, i.e. $\Leftrightarrow \lim\limits_{\substack{n \to \infty \\ x \in B(0_{E^n}; r)}} f\left(\dfrac{x}{2^n}\right) = f(0_{E^n}) \Leftrightarrow f$ is continuous in 0_{E^n}. By Lemma 2. f is continuous in E^n. The *case* $f(0_{E^n}) < 0$ *is analogous*. By induction we have

$$f\left(\frac{x}{2^n}\right) = -f(x) f(0_{E^n})^{1/2^n} |f(0_{E^n})|^{1-1/2^{n-1}}. \tag{5}$$

Passing to limit in (5), we have $\lim\limits_{\substack{n \to \infty \\ x \in B(0_{E^n}; r)}} f\left(\dfrac{x}{2^n}\right) = f(0_{E^n})$, i.e. f is continuous in 0_{E^n} and by Lemma 2. f is continuous in E^n.

Proposition 1. *Let $f : E^n \to \mathbf{R}$, $f(0_{E^n}) \ne 0$ be a solution of* (1). *If f is bounded in $B(0_{E^n}; r)$, then f is differentiable at 0_{E^n} and*

$$\frac{\partial f}{\partial \xi^k}(0_{E^n}) = \frac{f(0_{E^n})}{\xi^k} \log \frac{f(\xi^k e_k)}{f(0_{E^n})} \qquad k = 1, \ldots, n, \tag{6}$$

$$f(x) = \beta e^{\langle a, x\rangle}, \beta = f(0_{E^n}), a = \frac{1}{f(0^n_E)} \cdot df(0_{E^n}), \tag{7}$$

$$\frac{\partial f}{\partial \xi^k}(x) = \frac{f(x)}{f(0_{E^n})} \frac{\partial f}{\partial \xi^k}(0_{E^n}), \text{ for all } x \in E^n, \tag{8}$$

$$\frac{\partial^{|\alpha|} f}{(\partial \xi^1)^{\alpha_1} \ldots (\partial \xi^n)^{\alpha_n}}(x) = \left(\frac{f(x)}{f(0_{E^n})}\right)^{|\alpha|} \cdot \frac{\partial^{|\alpha|} f}{(\partial \xi^1)^{\alpha_1} \ldots (\partial \xi^n)^{\alpha_n}}(0_{E^n}), \tag{9}$$

where $x \in E^n$, $\alpha_k \in \mathbf{N}$ $(k = 1, \ldots, n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \ldots + \alpha_n \in \mathbf{N}$, i.e, $f \in C^\infty_{E^n}$

Proof.

$$\frac{\partial f}{\partial \xi^k}(0_{E^n}) = \lim_{\substack{t \to 0 \\ t \neq 0}} \frac{f(te_k) - f(0_{E^n})}{t} \tag{10}$$

is the derivative of f at 0_{E^n} along the unit vector e_k. Taking into account Lemma 3, (4) and

$$\lim_{n\to\infty} \frac{\left(\dfrac{f(\xi^k e_k)}{f(0_{E^n})}\right)^{1/2^n} - 1}{1/2^n} = \log \frac{f(\xi^k e_k)}{f(0_{E^n})}, \tag{11}$$

we obtain

$$\frac{f\left(\dfrac{\xi^k}{2^n} e_k\right) - f(0_{E^n})}{\xi^k/2^n} = \frac{f(0_{E^n})}{\xi^k} \frac{\dfrac{f\left(\dfrac{\xi^k}{2^n} e_k\right)}{f(0_{E^n})} - 1}{1/2^n}, \qquad \text{i.e.}$$

$$\frac{f\left(\dfrac{\xi^k}{2^n} e_k\right) - f(0_{E^n})}{\dfrac{\xi^k}{2^n}} = \frac{f(0_{E^n})}{\xi^k} \frac{\left(\dfrac{f(\xi^k e_k)}{f(0_{E^n})}\right)^{1/2^n} - 1}{\dfrac{1}{2^n}}$$

if $\xi^k \neq 0$, $f(0_{E^n}) > 0$

$$\lim_{\substack{n \to \infty \\ \xi^k \neq 0}} \frac{f\left(\dfrac{\xi^k}{2^n} e_k\right) - f(0_{E^n})}{\xi^k/2^n} = \frac{f(0_{E^n})}{\xi^k} \log\left(\frac{f(\xi^k e_k)}{f(0_{E^n})}\right) = \frac{\partial f}{\partial \xi^k}(0_{E^n}), \tag{12}$$

where $k = 1, \ldots, n$. From (6) we have

$$f(\xi^k e_k) = f(0_{E^n}) \exp\left(\frac{1}{f(0_{E^n})}\left(\xi^k \frac{\partial f}{\partial \xi^k}(0_{E^n})\right)\right),$$

$$f(x) = f\left(\sum_{k=1}^{n} \xi^k e_k\right) = f(0_{E^n}) \exp\left(\frac{1}{f(0_{E^n})} \sum_{k=1}^{n} \xi^k \frac{\partial f}{\partial \xi^k}(0_{E^n})\right) = \beta e^{\langle a,x\rangle}$$

i.e. (7). The differential $df(0_{E^n}) = \sum_{k=1}^{n} \frac{\partial f}{\partial \xi^k}(0_{E^n}) d\xi^k$ exists because the partial derivatives $\frac{\partial f}{\partial \xi^k}(0_{E^n})$ are continuous (see (6)), $\xi^k \neq 0$ and $f(\xi^k e_k)$ is continuous by Lemma 3, $\frac{f(x)}{f(0_{E^n})} > 0$. The case $f(0_{E^n}) < 0$ is analogoues. For (8)–(9) we have

$$\frac{f(x + te_k) - f(x)}{t} = \frac{1}{2f(-x)}\Big(f(te_k/2) + f(0_{E^n})\Big) \frac{f(te_k/2) - f(0_{E^n})}{t/2},$$

$$\lim_{\substack{t \to 0 \\ t \neq 0}} \frac{f(x+te_k)-f(x)}{t} = \frac{f(0_{E^n})}{f(x)} \frac{\partial f}{\partial \xi^k}(0_{E^n}) = \frac{f(x)}{f(0)} \frac{\partial f}{\partial \xi^k}(0_{E^n}), \text{ i.e. } (8)$$

and $\dfrac{\partial f}{\partial \xi^k}(x)$, $k = 1, \ldots, n$ are continuous. By induction successively using

$$\frac{\partial}{\partial \xi^k}(\,)(x) = \frac{f(x)}{f(0_{E^n})} \frac{\partial}{\partial \xi^k}(\,)(0_{E^n}),$$

we obtain (9).

The following results are almost evident.

3. Lemma 4. *If* $f : E^n \to \mathbf{R}$, $f(0_{E^n}) \neq 0$ *is a solution of* (1), *then*

(13) $$g = g(x) = \frac{f(x)}{f(0_{E^n})} : E^n \to \mathbf{R}$$

is a solution for Cauchy*'s multiplicative functional equation* [**1**]

(14) $$g(x+y) = g(x) \cdot g(y)$$

for all $x, y \in E^n$ *and conversely, if* $g : E^n \to \mathbf{R}$ *is a solution of* (14), *then*

(15) $$f(x) = \beta g(x),\ \beta = f(0_{E^n}) \neq 0$$

is a solution of (1).

Proposition 2. *By the same assumptions as in Proposition* 1. *the solution of* (14) *is*

(16) $$g(x) = e^{ax} = e^{\langle a,x \rangle} = e^{\alpha^1\xi^1 + \alpha^2\xi^2 + \cdots + \alpha^n\xi^n}$$

Lemma 5. *If* $f : E^n \to \mathbf{R}$, $f(0_{E^n}) \neq 0$ *is a solution of* (11), *then*

(17) $$h = h(x) = \log \frac{f(x)}{f(0_{E^n})} : E^n \to \mathbf{R}$$

is a solution for Cauchy*'s additive functional equation* [**1**]

(18) $$h(x+y) = h(x) + h(y)$$

for all $x, y \in E^n$ *and conversely, if* $h : E^n \to \mathbf{R}$ *is a solution of* (18), *then*

(19) $$f(x) = \beta e^{h(x)},\ \beta = f(0_{E^n}) \neq 0$$

is a solution of (1).

Proposition 3. *By the same assumptions as in Proposition* 1, *the solution of* (18) *is*

$$h(x) = a\,x = \langle a, x\rangle = \sum_{k=1}^{n} \alpha^k \xi^k. \tag{20}$$

Lemma 6. *If* $f : E^n \to \mathbf{R}$, $f(0_{E^n}) > 0$ *is a solution of* (1), *then*

$$\varphi = \varphi(x) = \log\, f(x) : E^n \to \mathbf{R} \tag{21}$$

is a solution for Jensen*'s functional equation* [**1**]

$$\varphi\left(\frac{x+y}{2}\right) = \frac{1}{2}\left(\varphi(x) + \varphi(y)\right) \tag{22}$$

and conversely, if $\varphi(x)$ *is a solution of* (22), *then*

$$f(x) = e^{\varphi(x)} \tag{23}$$

is a solution of (1).

Proposition 4. *If* $f : E^n \to \mathbf{R}$, $f(0_{E^n}) > 0$ *is bounded in* $B(0_{E^n}; r)$, *then the solution of* (22) *is*

$$\varphi(x) = ax + \gamma = \langle a, x\rangle + \gamma = \sum_{k=1}^{n} \alpha^k \xi^k + \gamma,\ \gamma = \log\, f(0_{E^n}) = \log\, \beta. \tag{24}$$

Lemma 7. *If* $f : E^n \to \mathbf{R}$, $f(0_{E^n}) \neq 0$ *is a solution of* (1), *then*

$$g = g(x) = \frac{f(x) + f(-x)}{2f(0_{E^n})},\ h = h(x) = \frac{f(x) - f(-x)}{2f(0_{E^n})} : E^n \to \mathbf{R} \tag{25}$$

verify

$$g(0_{E^n}) = 1,\ h(0_{E^n}) = 0,\ g(-x) = g(x),\ h(-x) = -h(x), \tag{26}$$

$$g(x)^2 - h(x)^2 = 1, \tag{27}$$

$$g(x)^2 + h(x)^2 = g(2x), \tag{28}$$

$$2h(x)g(x) = h(2x), \tag{29}$$

$$g(x+y) = g(x)g(y) + h(x)h(y), \tag{30}$$

$$h(x+y) = h(x)g(y) + h(y)g(x), \tag{31}$$

$$2g(x)^2 = 1 + g(2x),\ 2h(x)^2 = g(2x) - 1, \tag{32}$$

$$g(x+y) + g(x-y) = 2g(x)g(y),\ g(x+y) - g(x-y) = 2h(x)h(y) \tag{33}$$

and conversely, if $(g(x), h(y))$ *is a solution of* $(30) - (31)$, *then*

$$f(x) = \beta\,\big(g(x) + h(x)\big),\ \beta = f(0_{E^n}) \tag{34}$$

is a solution of (1).

Proof. From (1) and (25) results (26)-(29). For (30)-(31), we have

$$\begin{aligned} g(x+y) &= \frac{f\left(\frac{x+y}{2}\right)^2 + f\left(-\frac{x+y}{2}\right)^2}{2f(0_{E^n})} \\ &= 2\left(\frac{f\left(\frac{x+y}{2}\right) + f\left(-\frac{x+y}{2}\right)}{2f(0_{E^n})}\right)^2 - 1 = 2g\left(\frac{x+y}{2}\right)^2 - 1 \end{aligned}, \tag{35}$$

$$g(x)g(y) + h(x)h(y) = 2g\left(\frac{x+y}{2}\right)^2 - 1 \tag{36}$$

which imply (30). On the other side, from (31) we obtain

$$h(x+y) = 2g\left(\frac{x+y}{2}\right) h\left(\frac{x+y}{2}\right) \tag{37}$$

and

$$g(x)h(y) + g(y)h(x) = 2g\left(\frac{x+y}{2}\right) h\left(\frac{x+y}{2}\right) \tag{38}$$

whence it results (31). The relations (32)–(33) are consequences of previous relations. Conversely, from (34), we obtain

$$f(x)f(y) = f(0_{E^n})^2\big(g(x) + h(x)\big)\cdot\big(g(y) + h(y)\big) = f(0_{E^n})^2\big(g(x+y) + h(x+y)\big),$$

i.e

$$f(x)f(y) = f(0_{E^n})f(x+y) \tag{39}$$

From (26) and (31) results

$$g(x-y) = g(x)g(y) - h(x)h(y) \tag{40}$$

Now to demonstrate

$$f\left(\frac{x+y}{2}\right)^2 = f(0_{E^n})^2 f(x+y) \tag{41}$$

Using (34) and (35), (37), (27), we obtain

$$\begin{aligned} f\left(\frac{x+y}{2}\right)^2 &= f(0_{E^n})^2\left(g\left(\frac{x+y}{2}\right)+h\left(\frac{x+y}{2}\right)\right)^2 \\ &= f(0_{E^n})^2\Big(g(x+y)+h(x+y)\Big) = f(0_{E^n})f(x+y). \end{aligned}$$

From (39) and (41) we have (1).

Proposition 5. *Let f, $f(0_{E^n}) = 1$ be a solution of* (1)*. If f is bounded in $B(0_{E^n}; r)$, then the functions*

$$\begin{aligned} g = g(x) &= \frac{e^{ax}+e^{-ax}}{2} = \operatorname{ch} ax = \operatorname{ch}\left(\sum_{k=1}^{n}\alpha^k\xi^k\right), \\ h = h(x) &= \frac{e^{ax}-e^{-ax}}{2} = \operatorname{sh} ax = \operatorname{sh}\left(\sum_{k=1}^{n}\alpha^k\xi^k\right) : E^n \to \mathbf{R}, \end{aligned} \tag{42}$$

verify relations $(26)-(33)$.

The proof results from Lemma 7 and Proposition 1.

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