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A NOTE ON MOHR'S PAPER

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In this paper we establish the inequalities for maximum values of polynomials which have the zero as a simple root and as a multiple root on the unit circle.

Let $f(z)$ and $g(z)$ be two polynomials on the circle $|z|=1$ with degree $m \geq 1$ and $n \geq 1$ respectively. The formula $M_{fg} \geq \nu M_f M_g$ given in [**1**] has been strengthened in [**2**] by $M_{fg} > \nu_1 M_f M_g$, where $M_f = \max\limits_{|z|=1} |f(z)|$, $\nu = \sin^m \dfrac{\pi}{8m} \sin^n \dfrac{\pi}{8n}$ and $\nu_1 = \dfrac{1}{2^m} \, \dfrac{1}{2^n} > \nu$.

In this paper we show that, for a polynomial $f(z)$ of degree m having the point zero as a k-multiple root ($k < m$), and for a polynomial $g(z)$ of degree n of which the zero is an r-multiple root ($r < n$), we have

$$M_{fg} \geq \delta M_f M_g,$$

where $\delta = \dfrac{1}{2^{m-k}} \, \dfrac{1}{2^{n-r}} > \nu_1$.

1. Maximum values of polynomials which have the zero as a simple root on the unit circle

For $m, n, \ell > 1$ consider the following polynomials

$$(1) \qquad f(z) = z^m + \sum_{i=1}^{m-1} a_i z^i = z(z-\alpha_1)\cdots(z-\alpha_{m-1}),$$

$$(2) \qquad g(z) = z^n + \sum_{j=1}^{n-1} b_j z^j = z(z-\beta_1)\cdots(z-\beta_{n-1}),$$

$$(3) \qquad h(z) = z^\ell + \sum_{k=1}^{\ell-1} c_k z^k = z(z-1)\cdots(z-z_{\ell-1}) \quad \big(|z_1| \leq 1, \ldots, |z_{\ell-1}| \leq 1\big).$$

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From (3) it follows

$$M_h = \max_{|z|=1}\left\{\left|\frac{h(z)}{z^\ell}\right|\right\} = \max_{|z|=1}\left|\left(1-\frac{z_1}{z}\right)\cdots\left(1-\frac{z_{\ell-1}}{z}\right)\right|.$$

If we take $s(t) = (1-z_1t)\cdots(1-z_{\ell-1}t)$ with $t = 1/z$, we can rewrite M_h as follows:

$$M_h = \max_{|t|\le 1}|s(t)| \qquad \big(s(0)=1\big).$$

By the *maximum modulus principle* we have $M_h \ge 1$ and from definition of M_h, $M_h \le 2^{\ell-1}$.

By similar arguments, we obtain

$$M_f \le 2^{m-1}, \qquad M_g \le 2^{n-1}.$$

But in the case $z_1 = z_2 = \cdots = z_{\ell-1} = e^{i\theta_0}$ $(\theta_0 \in \mathbf{R})$, we have $M_h = 2^{\ell-1}$. Since $|\overline{\gamma}| > 1$ for $|\gamma| > 1$, $|1/\overline{\gamma}|$ is in the unit circle. Now consider as in [**2**]

$$(z-\gamma) = \left(z - \frac{1}{\overline{\gamma}}\right)\cdot\overline{\gamma}\cdot\frac{\gamma - z}{1-\overline{\gamma}z}, \qquad \left(\left|\frac{\gamma-z}{1-\overline{\gamma}z}\right| = 1, \ \ |z|=1\right).$$

In order to form the polynomials the ordering of roots as follows

$$\begin{array}{ll} 0,\alpha_1,\ldots,\alpha_{p-1}/\alpha_p,\ldots,\alpha_{m-1} & ;|\alpha_p|>1,\ldots|\alpha_{m-1}|>1,\\ 0,\beta_1,\ldots,\beta_{q-1}/\beta_q,\ldots,\beta_{n-1} & ;|\beta_q|>1,\ldots|\beta_{n-1}|>1,\end{array}$$

and write

$$F(z) = z(z-\alpha_1)\cdots(z-\alpha_{p-1})\left(z-\frac{1}{\overline{\alpha}_p}\right)\cdots\left(z-\frac{1}{\overline{\alpha}_{m-1}}\right), \tag{4}$$

$$G(z) = z(z-\beta_1)\cdots(z-\beta_{q-1})\left(z-\frac{1}{\overline{\beta}_q}\right)\cdots\left(z-\frac{1}{\overline{\beta}_{n-1}}\right). \tag{5}$$

Now we have the new forms of

$$f(z) = AF(z)\prod_{p}^{m-1}\left(\frac{\alpha_\mu - z}{1-\overline{\alpha}_\mu z}\right), \qquad g(z) = BG(z)\prod_{q}^{n-1}\left(\frac{\beta_\eta - z}{1-\overline{\beta}_\eta z}\right),$$

where $A = \overline{\alpha}_p\cdots\overline{\alpha}_{m-1}$, $B = \overline{\beta}_q\cdots\overline{\beta}_{n-1}$. Hence it is clear that

$$M_f = |A|M_F, \ \ M_g = |B|M_G, \ \ M_{fg} = |A|\,|B|M_{FG}.$$

Combining these last equalities we find

$$\frac{M_{fg}}{M_fM_g} = \frac{M_{FG}}{M_FM_G}. \tag{6}$$

On the other hand, the polynomials $F(z)$ and $G(z)$ are of the type (3). Then we can say $M_F \le 2^{m-1}$, $M_G \le 2^{n-1}$ and $M_{FG} \ge 1$. Thus, by (6) we will have

$$M_{fg} \ge \nu_2 M_fM_g, \tag{7}$$

where $\nu_2 = \frac{1}{2^{m-1}}\frac{1}{2^{n-1}}$. The fact that $\nu_2 > \nu_1$ is obvious.

Result. Let $f(z)$ and $g(z)$ be given as (1) and (2). Suppose that $f_1(z)$ and $g_1(z)$ are polynomials on the unit circle $D = \{z : |z| \leq 1,\ z \in \mathbf{C}\}$ of degrees $m-1$ and $n-1$, respectively, for which the zero point is not their root. Then we have

$$M_{f_1} M_{g_1} M_f M_g \leq {\nu_2}^{-2} M_{f_1 g_1} M_{fg}.$$

Proof. Since $f(z)$ and $g(z)$ are polynomials satisfying (7), we will have $\nu_2 = \frac{1}{2^{m-1}}\frac{1}{2^{n-1}}$. But at the same time for $f_1(z)$ and $g_1(z)$, $\nu_1 = \frac{1}{2^{m-1}}\frac{1}{2^{n-1}}$ holds in view of [**2**]. In other words, $\nu_1 = \nu_2$.

2. Maximum values of polynomials which have the zero as a multiple root on the unit circle

Theorem. *Let $f(z) = z^k(z-\alpha_1)\cdots(z-\alpha_{m-k})$ and $g(z) = z^r(z-\beta_1)\cdots(z-\beta_{n-r})$ be two polynomials in D. Then*

$$M_{fg} \geq \delta M_f M_g, \tag{8}$$

where $\delta = \frac{1}{2^{m-k}}\frac{1}{2^{n-r}}$.

Proof. Consider on D the polynomial $h(z) = z^k(z-z_1)\cdots(z-z_{w-k})$ of degree w. It is clear that we have $M_h \leq 2^{w-k}$, $M_f \leq 2^{m-k}$ and $M_g \leq 2^{n-r}$. Similar to (4) and (5), we form

$$\begin{aligned} F(z) &= z^k(z-\alpha_1)\cdots(z-\alpha_{p-1})\left(z-\frac{1}{\overline{\alpha}_p}\right)\cdots\left(z-\frac{1}{\overline{\alpha}_{m-k}}\right), \\ G(z) &= z^r(z-\beta_1)\cdots(z-\beta_{q-1})\left(z-\frac{1}{\overline{\beta}_q}\right)\cdots\left(z-\frac{1}{\overline{\beta}_{n-r}}\right). \end{aligned}$$

Then

$$f(z) = AF(z)\prod_p^{m-k}\left(\frac{\alpha_\mu - z}{1-\overline{\alpha}_\mu z}\right), \qquad g(z) = BG(z)\prod_q^{n-r}\left(\frac{\beta_\eta - z}{1-\overline{\beta}_\eta z}\right),$$

where $A = \overline{\alpha}_p \cdots \overline{\alpha}_{m-k}$, $B = \overline{\beta}_q \cdots \overline{\beta}_{n-r}$. Hence, we have on the circle $|z| = 1$

$$M_G = |A| M_F, \quad M_g = |B| M_G, \quad M_{fg} = |A|\,|B| M_{FG}.$$

Besides, one has $M_F \leq 2^{m-k}$, $M_G \leq 2^{n-r}$ and $M_{FG} \geq 1$. Thus, for $\delta = \frac{1}{2^{m-k}}\frac{1}{2^{n-r}}$ we obtain (8).

Corollary. *For δ to be equal to 1 it is necessary and sufficient that the polynomials $f(z)$ and $g(z)$ have the zero as m multiple root and $g(z)$ has n multiple root respectively.*

Proof. Immediate from Theorem.

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