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A BASIC SEPARATION ON A SET WITH APARTNESS

R. Milošević, D. A. Romano, M. Vinčić

This investigation is in constructive mathematics. In this short note we take into consideration a basic separation on a set X with an apartness which is a generalization of diversity relation on the power-set $P(X)$.

1. INTRODUCTION

Following Bishop [**1**] we regard the equality relation on a set X as conventional. A *diversity* relation on a set X is a binary relation $\neq$ satisfying the properties $\neg(x \neq x)$, $x \neq y \Rightarrow y \neq x$. Note that, by our implicit assumption of extensionality a diversity relation satisfies the condition

$$x \neq y \wedge y = z \Rightarrow x \neq z.$$

A diversity that also satisfies the condition

$$x \neq z \Rightarrow (\forall y)\,(x \neq y \vee y \neq z)$$

is called an *appartness.* Let $Y \subseteq X$ and $x \in X$. We define $x\#Y$ if and only if $(\forall y \in Y)\,(y \neq x)$ and $\overline{Y} = \{x \in X : x\#Y\}$. Equality and diversity relations on the power-set $\boldsymbol{P}(X)$ defined by

$$A = B \Leftrightarrow A \subseteq A \wedge B \subseteq A,$$
$$A \neq B \Leftrightarrow (\exists a \in A)\,(a\#B) \vee (\exists b \in B)\,(b\#A).$$

In this paper we shall give definition of the notion of basic separation. Further, we shall develop a theory of base and subbase of basic separation and we shall give some characteristics of these notions.

For all notions and notations, which will be used here, we refer to the books [**1, 3, 5**] and to the papers [**2, 4**].

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2. BASIC SEPARATIONS

With a following definition we shall introduce a notion of basic separation on a set X with apartness which is a generalization of diversity relation on the power-set $\boldsymbol{P}(X)$.

Definition 1. *A basic separation on a set X is a relation d on $\boldsymbol{P}(X)$ satisfies*

(d1) $d\#(A,A)$,

(d2) $AdB \Rightarrow BdA$,

(d3) $AdB \Rightarrow A \subseteq \overline{B}$,

(d4) $Ad(B \cup C) \Leftrightarrow AdB \wedge AdC$.

EXAMPLES

1° Let we define Ad_0B if and only if $A \subseteq \overline{B}$ and $d_0\#(A,A)$. Then we have:

(ii) $Ad_0B \Leftrightarrow A \subseteq \overline{B} \Rightarrow \overline{A} \supseteq \overline{\overline{B}} \wedge \overline{\overline{B}} \supseteq B \Rightarrow B \subseteq \overline{A} \Leftrightarrow Bd_0A$.

(iv) $Ad_0(B \cup C) \Leftrightarrow A \subseteq \overline{B \cup C} = \overline{B} \cup \overline{C} \Leftrightarrow A \subseteq \overline{B} \wedge A \subseteq \overline{C} \Leftrightarrow Ad_0B \wedge Ad_0C$.

2° If we define Ad_1B if and only if $\overline{\overline{A}} \subseteq \overline{B}$ and $d_1\#(A,A)$, we have

(ii) $Ad_1B \Leftrightarrow \overline{\overline{A}} \subseteq \overline{B} \Rightarrow \overline{\overline{B}} \subseteq \overline{\overline{\overline{A}}} = \overline{A} \Leftrightarrow Bd_1A$.

(iii) $Ad_1B \Leftrightarrow \overline{\overline{A}} \subseteq \overline{B} \Rightarrow \overline{\overline{A}} \subseteq \overline{B} \wedge A \subseteq \overline{\overline{A}} \Rightarrow A \subseteq \overline{B}$.

(iv) $Ad_1(B \cup C) \Leftrightarrow \overline{\overline{A}} \subseteq \overline{B \cup C} = \overline{B} \cap \overline{C} \Leftrightarrow \overline{\overline{A}} \subseteq \overline{B} \wedge \overline{\overline{A}} \subseteq \overline{C} \Leftrightarrow Ad_1B \wedge Ad_1C$.

Note that is in effect $Ad_1B \Rightarrow Ad_0B$.

Let d_a and d_b be two basic separation on a set X. If $Ad_bB \Rightarrow Ad_aB$, we say that d_a is *finer* that d_b and d_b is *coarser* than d_a.

We start with the following theorem:

Theorem 1. *Let d be a basic separation on a set X with appartness. Then*:

(1) $AdB \Rightarrow Ad_0B$.

(2) $(A \cup B)dC \Leftrightarrow AdC \wedge BdC$.

(3) $AdB \wedge A'' \subseteq A \wedge B'' \subseteq B \Rightarrow A''dB''$.

Proof.

(2)
$$\begin{aligned} (A \cup B)dC &\Leftrightarrow Cd(A \cup B) && \text{by (d2)} \\ &\Leftrightarrow CdA \wedge CdB && \text{by (d4)} \\ &\Leftrightarrow AdC \wedge BdC && \text{by (d2).} \end{aligned}$$

(3)
$$\begin{aligned} AdB \wedge A'' \subseteq A \wedge B'' \subseteq B &\Rightarrow AdB = (A \cup A'')dB \\ &\Rightarrow AdB \wedge A''dB && \text{by (d2)} \\ &\Rightarrow A''dB = A''d(B \cup B'') \\ &\Rightarrow A''dB \wedge A''dB'' && \text{by (d4)} \\ &\Rightarrow A''dB''. \quad \square. \end{aligned}$$

Corollary 1.1. *Let d be a nonempty basic separation on X. Then $Ad\emptyset$ and $\emptyset dB$ if AdB.*

Corollary 1.2. *Let d be a basic separation on X. Then $x \neq y$ if $\{x\}d\{y\}$ for x, y in X.*

Proof. Let x, y be arbitrary elements of X and let $\{x\}d\{y\}$. Then, by (d3), we have $\{x\} \subseteq \overline{\{y\}}$, i.e. $x\#\{y\}$. This means that $x \neq y$. □

Corollary 1.3. *Let d be a basic separation on X and $B \subseteq X$ and $x \in X$ such that $\{x\}dB$. Then $x\#B$.*

Proof. This immediately follows from (d3). □

We close this section with results which are related to family of basic separations.

Theorem 2. *Let $\{d_i : i \in I\}$ be a nonempty family of basic separations on a set X with apartness. Then the relation $d = \bigcap_{i \in I} d_i$ is a basic separation on X coarser than each d_i $(i \in I)$.*

Corollary 2.1. *The collection of all basic separations on set X forms a semilattice under the natural ordering.*

3. BASE OF BASIC SEPARATION

Basic separation may be characterised by its base. This notion is given in next definition.

Definition 2. *A base of a basic separation is a binary relation b on $\boldsymbol{P}(X)$ defined by axioms*:

(b1) $b\#(A, A)$,

(b2) $(A, B) \in b \Rightarrow (B, A) \in b$,

(b3) $(A, B) \in b \Rightarrow A \subseteq \overline{B}$,

(b4) $(A, B) \in b \wedge A'' \subseteq A \wedge B'' \subseteq B \Rightarrow (A'', B'') \in b$.

If b_1 and b_2 are two basic separations on a set X with apartness such that $(A, B) \in b_2 \Rightarrow (A, B) \in b_1$, we say that b_1 is finer than b_2 and b_2 is coarser than b_1.

Having a base of basic separation on a set X with apartness we can construct a basic separation on X. In the next theorem we shall give construction of that kind.

Theorem 3. *Let b be a base of a basic separation on a set X. Then the relation $d = d(b)$, defined by AdB if for even one finite cover $\{A_i : i \in I_n\}$ and $\{B_j : j \in I_m\}$ of A and B respectively, we have $(\forall i \in I_n)(\forall j \in I_m)\big((A_i, B_j) \in b\big)$, is a basic separation on X.*

Proof.

(ii) $$AdB \Leftrightarrow A = \bigcup_{i=1}^{n} A_i \wedge B = \bigcup_{j=1}^{m} B_j \ \wedge\ (\forall i)\,(\forall j)\,\big((A_i, B_j) \in b\big)$$
$$\Leftrightarrow A = \bigcup_{i=1}^{n} A_i \wedge B = \bigcup_{j=1}^{m} B_j \ \wedge\ (\forall j)\,(\forall i)\,\big((B_j, A_i) \in b\big)$$
$$\Leftrightarrow BdA.$$

(iii) $$AdB \Leftrightarrow A = \bigcup_{i=1}^{n} A_i \wedge B = \bigcup_{j=1}^{m} B_j \ \wedge\ (\forall i)\,(\forall j)\,\big((A_i, B_j) \in b\big)$$
$$\Rightarrow\ A = \bigcup_{i=1}^{n} A_i \wedge B = \bigcup_{j=1}^{m} B_j \ \wedge\ (\forall i)\,(\forall j)\,(A_i \subseteq \overline{B}_j)$$
$$\Rightarrow\ A = \bigcup_{i=1}^{n} A_i \subseteq \bigcap_{j=1}^{m} \overline{B}_j = \overline{\bigcup_{j=1}^{m} B_j} = \overline{B}.$$

(iv) Let AdB and AdC and let $A = \bigcup_{i=1}^{n} A'_i$, $B = \bigcup_{j=1}^{m} B_j$ $\big((\exists j)\,(B_j \neq \emptyset)\big)$ such that $(\forall i)\,(\forall j)\,\big((A'_i, B_j) \in b\big)$. Also suppose that $A = \bigcup_{k=1}^{p} A''_k$, $C = \bigcup_{s=1}^{q} C_s$ $\big((\exists s)\,(C_s \neq \emptyset)\big)$ such that $(\forall k)\,(\forall s)\big((A''_k, C_s) \in b\big)$. Let we define

$$A_{ij} = A'_i \cup A''_j \qquad (i = 1, \ldots, n; j = 1, \ldots, p),$$
$$D_t = B_t \qquad (t = 1, \ldots, m),$$
$$D_{m+r} = C_r \qquad (r = 1, \ldots, q).$$

Then, by (b4), we have $(\forall x)\,(\forall y)\,\big((A_x, D_y) \in b\big)$, i.e. we have $Ad(B \cup C)$.

Conversely, let $Ad(B \cup C)$. If $A = \bigcup_{i=1}^{n} A_i$ $\big((\exists i)\,(A_i \neq \emptyset)\big)$ and $B \cup C = \bigcup_{j=1}^{m} D_j$ with $(\forall i)\,(\forall j)\,\big((A_i, D_j) \in b\big)$, then for $B = \bigcup_{j=1}^{m} (B \cap D_j) = \bigcup_{j=1}^{m} B_j$ and $C = \bigcup_{j=1}^{m} (C \cap D_j) = \bigcup_{j=1}^{m} C_j$ holds $(\forall i)\,(\forall j)\,\big((A_i, B_j) \in b\big)$ and $(\forall i)\,(\forall j)\,\big((A_i, C_j) \in b\big)$, i.e. hold AdB and AdC. □

Corollary 3.1. *Let b be a base of a basic separation on a set X with apartness. Then $(A, B) \in b \Rightarrow Ad(b)B$.*

We end this section with the following results.

Theorem 4. *Let $\{d_i : i \in I\}$ be a nonempty family of basic separations on a set X with apartness. Then the relation $b = \bigcup_{i \in I} d_i$ is a base of a basic separation on X.*

Corollary 4.1. *Let $\{d_i : i \in I\}$ be a nonempty family of basic separations on a set X with apartness. Then the relation $d = d\left(\bigcup_{i \in I} d_i\right)$ is a basic separation on X coarser than each d_i.*

Corollary 4.2. *The collection of all basic separations on a set X with apartness forms a lattice under the natural ordering.*

4. SUBBASE OF BASIC SEPARATION

There is a possibility for reducing of number of base axioms. In that case we shall get a new notion. In the following definition we introduce that notion.

Definition 3. *A subbase of a basic separation on a set X with apartness is a binary relation s on $\boldsymbol{P}(X)$ such that*

(s1) $(A, A) \# s,$

(s2) $(A, B) \in s \Rightarrow A \subseteq \overline{B},$

(s3) $(A, B) \in s \Rightarrow (B, A) \in s.$

The next theorem describes the way how we construct a base of basic separation on X with apartness if it is given a subbase.

Theorem 5. *If s is subbase of a basic separation, then the relation $b = b(s)$ on $\boldsymbol{P}(X)$, defined by*

$$(A, B) \in b \Leftrightarrow (\exists A' \supseteq A)\,(\exists B' \supseteq B)\,\big((A', B') \in s\big),$$

is a base of a basic separation on X.

Proof.

(i) $$\begin{aligned}(A, B) \in b &\Leftrightarrow (\exists A' \supseteq A)\,(\exists B' \supseteq B)\,\big((A', B') \in s\big)\\ &\Leftrightarrow (\exists B' \supseteq B)\,(\exists A' \supseteq A)\,\big((B', A') \in s\big)\\ &\Leftrightarrow (B, A) \in b.\end{aligned}$$

(ii) $$\begin{aligned}(A, B) \in b &\Leftrightarrow (\exists A' \supseteq A)\,(\exists B' \supseteq B)\,\big((A', B') \in s\big)\\ &\Rightarrow A' \subseteq \overline{B'} \wedge A \subseteq A' \wedge \overline{B'} \subseteq \overline{B}\\ &\Rightarrow A \subseteq \overline{B}.\end{aligned}$$

(iii) $$\begin{aligned}&(A, B) \in b \wedge A'' \subseteq A \wedge B'' \subseteq B \Leftrightarrow\\ &(\exists A' \supseteq A)\,(\exists B' \supseteq B)\,\big((A', B') \in s\big) \wedge A'' \subseteq A \wedge B'' \subseteq B \Rightarrow\\ &(\exists A' \supseteq A \supseteq A'')\,(\exists B' \supseteq B \supseteq B'')\,\big((A', B') \in s\big) \Rightarrow\\ &(A'', B'') \in b. \quad \square\end{aligned}$$

Corollary 5.1. *Let s be a subbase of a basic separation on a set X with apartness. Then $(A, B) \in s \Rightarrow (A, B) \in b(s)$.*

5. d-STRONGLY EXTENSIONAL FUNCTION

In this section we define and discuss *d-strongly extensional* function.

Definition 4. *Let d_X and d_Y be basic separations on sets X and Y respectively and let $f : X \to Y$ be a total function. We say that the function f is d-strongly extensional function if and only if $f(A)d_Y f(B)$ implies $Ad_X B$ for every A, B in $\boldsymbol{P}(X)$.*

We shall give a criterion to be d-strongly extensional function $f : X \to Y$ of spaces with basic separations by subbase of d_Y.

Theorem 6. *Let (X, d_X) and (Y, d_Y) be two spaces with basic separations and let s_Y be a subbase for d_Y. A function $f : X \to Y$ is d-strongly extensional if and only if $(C, D) \in s_Y$ implies $f^{-1}(C)d_X f^{-1}(D)$.*

Proof. 1° Let $f : X \to Y$ be d-strongly extensional function, and let C and D be subsets of Y such that $(C, D) \in s_Y$. Then $(C, D) \in b(s_Y)$ and $Cd_Y D$. As $ff^{-1}(C) \subseteq C$ and $ff^{-1}(D) \subseteq D$, we have that $ff^{-1}(C)d_Y ff^{-1}(D)$ implies $f^{-1}(C)d_X f^{-1}(D)$.

2° Let A and B be subsets of X such that $f(A)d_Y f(B)$. Then

$$
\begin{aligned}
&f(A) = \bigcup C_i \wedge f(B) = \bigcup D_j \wedge (\forall i)(\forall j)\big((C_i, D_j) \in b(s_Y)\big) \Leftrightarrow \\
&f(A) = \bigcup C_i \wedge f(B) = \bigcup D_j \wedge (\forall i)(\forall j)(\exists C_i' \supseteq C_i)(\exists D_j' \supseteq D_j)\big((C_i', D_j') \in s_Y\big) \Rightarrow \\
&f(A) = \bigcup C_i \wedge f(B) = \bigcup D_j \wedge (\forall i)(\forall j)(\exists C_i')(\exists D_j')\big(f^{-1}(C_i')d_X f^{-1}(D_j')\big) \Rightarrow \\
&f(A) = \bigcup C_i \wedge f(B) = \bigcup D_j \wedge (\forall i)(\forall j)\big(f^{-1}(C_i)d_X f^{-1}(D_j)\big) \Rightarrow \\
&f(A) = \bigcup C_i \wedge f(B) = \bigcup D_j \wedge \bigcup f^{-1}(C_i)\, d_X \bigcup f^{-1}(D_j) \Leftrightarrow \\
&f(A) = \bigcup C_i \wedge f(B) = \bigcup D_j \wedge f^{-1}\big(\bigcup C_i\big)d_X f^{-1}\big(\bigcup D_j\big) \Leftrightarrow \\
&f^{-1}\big(f(A)\big)d_X f^{-1}\big(f(B)\big) \wedge A \subseteq f^{-1}\big(f(A)\big) \wedge B \subseteq f^{-1}\big(f(B)\big) \Rightarrow \\
&Ad_X B. \quad \square
\end{aligned}
$$

The next theorem is an application of the theorem 6.

Theorem 7. *Let $\boldsymbol{F}$ be a nonempty family, each $f \in \boldsymbol{F}$ being a strongly extensional function on X to space (Y_f, d_f) with basic separation d_f. Then the relation b on $\boldsymbol{P}(X)$, defined by*

$$(A, B) \in b \Leftrightarrow (\exists f \in \boldsymbol{F})\big(f(A)d_f f(B)\big),$$

is a base of a basic separation on X. Further, each member of $\boldsymbol{F}$ is d-strongly extensional.

Proof.

$$
\begin{aligned}
(A, B) \in b &\Leftrightarrow (\exists f \in \boldsymbol{F})\big(f(A)d_f f(B)\big) \\
&\Leftrightarrow (\exists f \in \boldsymbol{F})\big(f(B)d_f f(B)\big) \\
&\Leftrightarrow (B, A) \in b.
\end{aligned}
$$

As

$$\begin{aligned}(A,B)\in b &\Leftrightarrow (\exists f\in \boldsymbol{F})\,\big(f(A)d_f f(B)\big)\\ &\Rightarrow (\exists f\in \boldsymbol{F})\,\big(f(A)\subseteq \overline{f(B)}\,\big),\end{aligned}$$

we have

$$\begin{aligned}a\in A &\Rightarrow f(a)\in f(A)\subseteq \overline{f(B)}\\ &\Rightarrow (\forall x\in B)\,\big(f(a)\neq f(x)\big)\\ &\Rightarrow (\forall x\in B)\,(a\neq x)\\ &\Leftrightarrow a\in \overline{B}.\end{aligned}$$

So, $A\subseteq \overline{B}$.

$$\begin{aligned}(A,B)\in b \wedge A''\subseteq A\wedge B''\subseteq B &\Rightarrow (\exists f\in F)\,\big(f(A)d_f f(B)\big)\\ &\Rightarrow (\exists f\in F)\,\big(f(A'')d_f f(B'')\big)\\ &\Rightarrow (A'',B'')\in b. \qquad \square\end{aligned}$$

Corollary 7.1. *Let $\langle (X_i,d_i)\rangle_{i=1}^n$ be a finite family of spaces with basic separations. Then the relation b on $\prod_{i=1}^n X_i$, defined by*

$$(A,B)\in b \Leftrightarrow (\exists p_k)\,\big(p_k(A)d_k p_k(B)\big),$$

where $p_k : \prod_{i=1}^n X_i \to X_k$ is a projection on X_k, is a base of basic separation on $\prod_{i=1}^n X_i$.

6. BASIC UNIFORMITY

In this section we study basic uniformities associated with basic separations.

Definition 5. ([**2**]) *A basic uniformity of X is a subfamily $\boldsymbol{U}$ of $\boldsymbol{P}(X^2)$ such that:*

(1) $(\forall R\in \boldsymbol{U})\,(I_X\subseteq R)$,

(2) $(\forall R\in \boldsymbol{U})\,(R^{-1}\in \boldsymbol{U})$,

(3) $(\forall R,S\in \boldsymbol{U})\,\big(\forall A\in \boldsymbol{P}(X)\big)\,(\exists T\in \boldsymbol{U})\,\big(T(A)\subseteq R(A)\cap S(A)\big)$,

(4) $(\forall R\in \boldsymbol{U})\,\big(R\subseteq S\wedge S\in \boldsymbol{P}(X^2)\Rightarrow S\in \boldsymbol{U}\big)$.

Subfamily $\boldsymbol{B}$ of $\boldsymbol{U}$ is called a base for a basic uniformity $\boldsymbol{U}$ if and only if for each element R of $\boldsymbol{U}$ there exists an element S of $\boldsymbol{B}$ such that $S\subseteq R$.

Subfamily $\boldsymbol{S}$ of $\boldsymbol{U}$ is a subbase for a basic uniformity $\boldsymbol{U}$ on X if and only if the family $\boldsymbol{B}$ of all finite intersections of elements of $\boldsymbol{S}$ is a base for $\boldsymbol{U}$.

Theorem 8. *Let $\boldsymbol{U}$ be a basic uniformity on X. Then the relation $s = s(\boldsymbol{U})$ on X, defined by*

$$AsB \Leftrightarrow (\exists R \in \boldsymbol{U})\,\big(R(A) \subseteq \overline{B}\,\big) \;\wedge\; (\exists S \in \boldsymbol{U})\,\big(S(B) \subseteq \overline{A}\,\big),$$

is a subbase of basic separation on X.

Proof.

(i) $AsB \Rightarrow (\exists R \in \boldsymbol{U})\,\big(R(A) \subseteq \overline{B}\,\big)$
$\Rightarrow A \subset \overline{B}$ by (1).

(ii) $AsB \Leftrightarrow (\exists R \in \boldsymbol{U})\,\big(R(A) \subseteq \overline{B}\,\big) \;\wedge\; (\exists S \in \boldsymbol{U})\,\big(S(B) \subseteq \overline{A}\,\big)$
$\Leftrightarrow BsA.$ □

Note that for the subbase $s = s(\boldsymbol{U})$ also holds:

$$\begin{aligned} AsB \wedge AsC \;&\Rightarrow\; (\exists R \in \boldsymbol{U})\,\big(R(A) \subset \overline{B}\,\big) \;\wedge\; (\exists S \in \boldsymbol{U})\,\big(S(A) \subseteq \overline{C}\,\big) \\ &\Rightarrow\; (\exists T \in \boldsymbol{U})\,\big(T(A) \subseteq R(A) \cap S(A) \subseteq \overline{B} \cap \overline{C} = \overline{B \cup C}\,\big). \end{aligned}$$

Theorem 9. *Let s be a subbase for a basic separation on a set X with apartness. Then the family*

$$S = S(s) = \{\,\overline{A \times B \cup B \times A} :\; A, B \in \boldsymbol{P}(X) \;\wedge\; AsB\}$$

is a subbase for a basic uniformity on X.

Proof. It is clearly that $\big(\overline{A \times B \cup B \times A}\big)^{-1} = \overline{A \times B \cup B \times A}$ if AsB. Let x be an element of X and let (u, v) be an arbitrary element of $A \times B \cup B \times A$, where $A, B \in \boldsymbol{P}(A)$ and AsB. Then $(u, v) \in A \times B$ or $(u, v) \in B \times A$. Then $u \neq v$ because $AsB \Rightarrow A \subseteq \overline{B}$. Thus $u \neq x \vee x \neq v$, i.e. $(x, x) \neq (u, v)$. So, $I_X \subseteq \overline{A \times B \cup B \times A}$. □

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Banja Luka University,
Faculty of Philosophy,
Department of Mathematics and Informatics,
78000 Banja Luka, Bana Lazarevića 1,
Republic of Srpska
daniel@urcbl.bl.ac.yu

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