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ABOUT THE EQUIVALENCE OF SOME FUNCTIONAL EQUATIONS

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The purpose ot this paper is to demonstrate the equivalence of Lobachevsky's functional equation

$$f(x)f(y) = f\left(\frac{x+y}{2}\right)^2 \qquad (f : \mathbf{R} \to \mathbf{R}) \tag{1}$$

with functional equations

$$f(x)^p f(y)^q = f\left(\frac{px+qy}{p+q}\right)^{p+q} \qquad (p, q \in \mathbf{R},\ p+q \neq 0) \tag{2}$$

and

$$\prod_{i=1}^{n} f(x_i)^{p_i} = f\left(\frac{\sum\limits_{i=1}^{n} p_i x_i}{\sum\limits_{i=1}^{n} p_i}\right)^{\sum\limits_{i=1}^{n} p_i} \qquad \left(p_i \in \mathbf{R},\ \sum_{i=1}^{n} p_i \neq 0.\right) \tag{3}$$

0. The following properties of Lobachevsky's functional equation are known [**1, 3**]:

a) Functional equation (1) is equivalent with

$$f(x+y)f(x-y) = f(x)^2; \tag{4}$$

b) For every solution f of (1) we have: $f > 0,\ x \in \mathbf{R}$ if $f(0) > 0$; $f < 0$ if $f(0) < 0$ and $f = 0$ if $f(0) = 0$;

c)

$$f(x)f(-x) = f(0)^2; \tag{5}$$

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d) The most general solution of (1) is

$$f(x) = f(0)g(x), \tag{6}$$

where $g : \mathbf{R} \to \mathbf{R}$ is the most general solution of CAUCHY's multiplicative functional equation

$$g(x+y) = g(x)g(y); \tag{7}$$

e) Let f, $f(0) \neq 0$ be a solution of (1). The function f is continuous on $\mathbf{R}$ if and only if f is continuous in zero;

f) Let f, $f(0) \neq 0$ be a solution of (1). If f is bounded on a neighbourhood $(-\varepsilon, \varepsilon)$ of zero, then f is continuos on $\mathbf{R}$.

1. Lemma 1. *If* $f : \mathbf{R} \to \mathbf{R}$, $f(0) \neq 0$ *is solution of* (1), *then* f *is solution of functional equation*

$$f(x)^m f(y)^n = f\left(\frac{mx+ny}{m+n}\right)^{m+n}, \qquad (m, n \in \mathbf{N}^*). \tag{8}$$

Proof. From (4) we successively obtain

$$f(2x)f(0) = f(x)^2; \qquad f(3x)f(x)f(-x) = f(x)^3,$$

hence $f(3x)f(0)^2 = f(x)^3$. We assume

$$f(mx)f(0)^{m-1} = f(x)^m \qquad (m > 3,\ m \in \mathbf{N}). \tag{9}$$

We have (see (6), (7), (9))

$$\begin{aligned} f\big((m+1)x\big)f(0)^m &= g(mx+x)f(0)^{m+1} = g(mx)g(x)f(0)^{m+1} \\ &= \frac{f(mx)}{f(0)}\,\frac{f(x)}{f(0)}\,f(0)^{m+1} = f(mx)f(x)f(0)^{m-1} \\ &= \frac{f(x)^m}{f(0)^{m-1}}\,f(x)f(0)^{m-1} = f(x)^{m+1}, \end{aligned}$$

hence

$$f\big((m+1)x\big)f(0)^m = f(x)^{m+1}.$$

Taking in account (6), (7), (9), the left-hand side of (8) becomes

$$\begin{aligned} f(x)^m f(y)^n &= f(mx)f(ny)f(0)^{m+n-2} = \frac{f(mx)}{f(0)}\,\frac{f(nx)}{f(0)}\,f(0)^{m+n} \\ &= g(mx)g(ny)f(0)^{m+n} = g(mx+ny)f(0)^{m+n} \\ &= f(mx+ny)f(0)^{m+n-1} \end{aligned}$$

and the right-hand side of (8) becomes:

$$f\left(\frac{mx+ny}{m+n}\right)^{m+n}=f\left((m+n)\,\frac{mx+ny}{m+n}\right)f(0)^{m+n-1}=f(mx+ny)f(0)^{m+n-1},$$

which implies (8).

2. Lemma 2. *If $f:\mathbf{R}\to\mathbf{R}$, $f(0)\neq 0$ is the solution of (1), then f is solution of functional equation*

$$f(x)^k f(y)^\ell = f\left(\frac{kx+\ell y}{k+\ell}\right)^{k+\ell};\qquad (k,\ell\in\mathbf{Z},\ k+1\neq 0). \tag{10}$$

Proof. From (5) and (9) results

$$f(-mx)=\frac{f(0)^2}{f(mx)}=f(x)^{-m}f(0)^{m+1},$$

hence

$$f(kx)f(0)^{k-1}=f(x)^k \qquad (k\in\mathbf{Z}).$$

We have

$$f(x)^k f(y)^\ell = f(kx)f(\ell y)f(0)^{k+\ell-2}=f(kx+\ell y)f(0)^{k+\ell-1}$$

and

$$f\left(\frac{kx+\ell y}{k+\ell}\right)^{k+\ell}=f\left((k+\ell)\,\frac{kx+\ell y}{k+\ell}\right)f(0)^{k+\ell-1}=f(kx+\ell y)f(0)^{k+\ell-1},$$

which proves that (10) is true.

3. Lema 3. *If $f:\mathbf{R}\to\mathbf{R}$, $f(0)>0$ is a solution of (1), then f is also a solution of functional equation*

$$f(x)^r f(y)^s = f\left(\frac{rx+sy}{r+s}\right)^{r+s} \qquad (r,s\in\mathbf{Q},\ r+s\neq 0). \tag{11}$$

Proof. We have (see b. (9))

$$f(x)=f\left(n\,\frac{x}{n}\right)=\frac{f(x/n)^n}{f(0)^{n-1}}\text{, i.e.}$$

$$f\left(\frac{1}{n}\,x\right)=f(0)^{1-(1/n)}f(x)^{1/n}\text{, and}$$

$$f\left(\frac{m}{n}\,x\right)f(0)^{(m/n)-1}=f(x)^{m/n}.$$

If $r=m/n,\ s=m_1/n_1\ \ (n,n_1\in\mathbf{N}^*,\ m,m_1\in\mathbf{Z})$, we obtain

$$\begin{aligned} f(x)^r f(y)^s &= f(x)^{m/n} f(y)^{m_1/n_1} = f\left(\frac{m}{n}\,x\right) f\left(\frac{m_1}{n_1}\,y\right) f(0)^{(m/n)+(m_1/n_1)-2} \\ &= g\left(\frac{m}{n}\,x+\frac{m_1}{n_1}\,y\right) f(0)^{(m/n)+(m_1/n_1)} = g(rx+sy)f(0)^{r+s} \\ &= f(rx+sy)f(0)^{r+s-1} \end{aligned}$$

and

$$f\left(\frac{rx+sy}{r+s}\right)^{r+s} = f\left((r+s)\,\frac{rx+sy}{r+s}\right) f(0)^{r+s-1}. \tag{11}$$

4. Lemma 4. *If* $f:\mathbf{R}\to\mathbf{R}$, $f(0)>0$ *and* f *is bounded on a neighbourhood* $(-\varepsilon,\varepsilon)$ *of zero, is a solution of* (1), *then* f *is a solution of functional equation* (2).
Proof. Let $(r_n)_{n\in\mathbf{N}}$, $(s_n)_{n\in\mathbf{N}}$ two sequences,

$$r_n, s_n\in\mathbf{Q},\ r_n+s_n\neq 0,\ \lim_{n\to+\infty} r_n=p,\ \lim_{n\to+\infty} s_n=q;\ p+q\neq 0\ (p,q\in\mathbf{R}\setminus\mathbf{Q}).$$

We have

$$f(x)^{r_n} f(y)^{s_n} = f\left(\frac{r_n x+s_n y}{r_n+s_n}\right)^{r_n+s_n}. \tag{12}$$

Taking into account b), e) and f) and passing to $\lim\limits_{n\to+\infty}$ in (12) we obtain functional equation (2).

Proposition 1. *Let* $f:\mathbf{R}\to\mathbf{R}$, $f(0)>0$ *is bounded on a neighbourhood* $(-\varepsilon,\varepsilon)$ *of zero, then Lobachevsky's functional equation* (1) *is equivalent with equation* (2).
Proof. Every solution of (1) (which verify the assumptions) is solution of (2) (Lemma 4). Reciprocally, every solution of (2) for $p=q=1$ is also solution for (1).

Proposition 2. *In the same assumptions as in Proposition* 1 *the solution of* (1) *is a convex function, i.e.*

$$\alpha f(x)+\beta f(y)\geq f(\alpha x+\beta y) \qquad (\alpha,\beta>0,\ \alpha+\beta=1).$$

The proof results from inequality [**2**]

$$a^\alpha b^\beta \leq \alpha a+\beta b \qquad (a,b,\alpha,\beta>0,\ \alpha+\beta=1)$$

anf from (2)

5. Lemma 5. *If* $f:\mathbf{R}\to\mathbf{R}$, $f(0)>0$, f *is bounded on small neighbourhood* $(-\varepsilon,\varepsilon)$ *of zero is solution of* (1), *then* f *is solution of* (3).

The proof results by mathematical induction. For $n = 2$, functional equation (3) becomes (2) and by Lemma 4 is true. We suppose that (3) is verified for $n > 2$ and results that (3) is true for $n + 1$.

Proposition 3. *Under the same assumptions as in Lemma* 5, *Lobachevski's functional equation* (1) *is equivalent with* (3).

The proof is similar with the proof of Proposition 1.

Proposition 4. *Under the same assumptions as in Lemma* 5, *we have*

$$\sum_{i=1}^{n} \alpha_i f(x_i) \geq f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \qquad (\alpha_i > 0,\ \sum_{i=1}^{n} \alpha_i = 1.$$

The proof results from equality [**2**]

$$\prod_{i=1}^{n} a_i^{\alpha_i} \leq \sum_{i=1}^{n} \alpha_i a_i \qquad \left(a_i, \alpha_i > 0,\ \sum_{i=1}^{n} \alpha_i = 1\right)$$

and from functional equation (3).

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