UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. 8 (1997), 16–23.

A METHOD OF CONSTRUCTING INEQUALITIES ABOUT e^x

QI Feng

Dedicated to the loving memory of my mother

We prove some inequalilies for exponential function $x \mapsto e^x$ and a combinatorial inequality, and present a method of constructing inequalities.

1. INTRODUCTION

From the TAYLOR's expansion

$$e^x = \sum_{k=0}^{\infty} x^k/k! \qquad (x \in \mathbf{R}) \tag{1}$$

it follows that, if n is an odd number,

$$e^x \geq 1 + \sum_{k=1}^{n} x^k/k! \qquad (x \in \mathbf{R}). \tag{2}$$

When n is an even number, (2) also holds for $x \geq 0$, however, (2) is reversed for $x \leq 0$. From this one could verify that the equation $\sum\limits_{k=0}^{2n} x^k/k! = 0$ has no real root [**1**, **4**, **5**, **6**].

Introducing the notation $S_n(x) = \sum\limits_{k=0}^{n} x^k/k!$ $(n \geq 0)$, we have

$$e^x - S_n(x) \leq \frac{xe^x}{n} \qquad (x \geq 0). \tag{3}$$

[0]1991 Mathematics Subject Classification: 26D20, 05D20

Let b be a given positive number. Then

$$\frac{x^{n+1}}{(n+1)!} \le e^x - S_n(x) \le \frac{x^{n+1}e^b}{(n+1)!} \qquad (x \in [0,b]). \tag{4}$$

For $x \in [0, n+1)$, the right side of (4) could be improved to [**4**, pp.358, **5** and **6**, pp. 269]

$$e^x - S_n(x) \le \frac{x^{n+1}}{(n-x+1)n!}. \tag{5}$$

Other inequalities about exponential function e^x may be found in [**1**, **2**, **4**, **5**, **6**].

In this article we prove some new inequalities for the exponential function and a combinatorial inequality. These results generalize and improve related inequalities in references by constructing the ancillary function, for example

$$e^x \le S_n(x) + \alpha_n x^{n+1} \qquad (x \in [0,b]), \tag{6}$$

where $\alpha_n = (\alpha_{n-1} - 1/n!)/b$, $\alpha_{-1} = e^b$, $n \ge 0$, which can not be replaced by smaller constants. So the inequality (6) is sharp.

Inequality (5) could be obtained by integrating (6) on both sides over $[0,b]$, hence (6) is better than (5) in this sense.

It is also noted that the method of the article could yield more general results and be applied to other questions, cf. [**3**, **7**–**9**].

2. LEMMAS

Lemma 1.

$$e^x - S_n(x) \ge \frac{n+2-(n+1)x}{(n+2)!}\, x^{n+1}e^x \qquad (x \in [0,+\infty)). \tag{7}$$

Proof. Suppose $h(x) = (n+1)x^{n+2}e^x - (n+2)x^{n+1}e^x + (n+2)!(e^x - S_n(x))$, $x \ge 0$. From (1) we obtain

$$\begin{aligned} h(x) &= (n+1)\sum_{k=0}^{\infty}\frac{x^{n+k+2}}{k!} - (n+2)\sum_{k=0}^{\infty}\frac{x^{n+k+1}}{k!} + (n+2)!\sum_{k=n+1}^{\infty}\frac{x^k}{k!} \\ &= (n+1)\sum_{k=0}^{\infty}\frac{x^{n+k+2}}{k!} - (n+2)\sum_{k=0}^{\infty}\frac{x^{n+k+2}}{(k+1)!} + (n+2)!\sum_{k=0}^{\infty}\frac{x^{n+k+2}}{(n+k+2)!} \\ &= \sum_{k=1}^{\infty}\left(\frac{k(n+1)-1}{(k+1)!} + \frac{(n+2)!}{(n+k+2)!}\right)x^{n+k+2} \ge 0 \quad (x \in [0,+\infty)). \text{ QED.} \end{aligned}$$

Lemma 2. $e^x - S_n(x) \le \dfrac{n+1+e^x}{(n+2)!}x^{n+1} \le \dfrac{e^x}{(n+1)!}x^{n+1} \qquad (x \ge 0).$

Proof. Set $\varphi(x)=(n+2)!\big(e^x-S_n(x)\big)-(n+1)x^{n+1}-x^{n+1}e^x$, then

$$\begin{aligned}\varphi^{(k)}(x) &= (n+2)!\big(e^x-S_{n-k}(x)\big)-\frac{(n+1)(n+1)!}{(n-k+1)!}\,x^{n-k+1}\\ &\quad -\sum_{i=0}^{k}\frac{(n+1)!}{(n-i+1)!}\,C_k^i x^{n-i+1}e^x,\\ \varphi^{(n+2)}(x) &= \left((n+2)!-\sum_{i=0}^{n+1}\frac{(n+1)!}{(n-i+1)!}\,C_{n+2}^i\,x^{n-i+1}\right)e^x \qquad (x\ge 0),\end{aligned}$$

where $1\le k\le n+1$, $\varphi^{(k)}(0)=0$. Since $\varphi^{(n+2)}(x)$ is a decreasing function and $\varphi^{(n+2)}(0)=0$, thus $\varphi^{(n+2)}(x)\le 0$, $\varphi^{(k)}(x)\le 0$, and $\varphi(x)\le 0$, $x\ge 0$. QED.

Lemma 3. *For $x\in[0,+\infty), 0\le k\le n$, we have*

$$\big(e^x-S_{n-k}(x)\big)x^k+\frac{k}{(n-k+2)!}\,x^{n+1}\ge\frac{(n+2)!}{(n-k+2)!}\big(e^x-S_n(x)\big). \tag{9}$$

Proof. Define a function $G:[0,+\infty)\to(-\infty,+\infty)$, by

$$G(x)=x^ke^x-x^kS_{n-k}(x)+\frac{k}{(n-k+2)!}\,x^{n+1}-\frac{(n+2)!}{(n-k+2)!}\big(e^x-S_n(x)\big).$$

For $0\le j\le k$, $1\le m\le n-k$, computing directly yields

$$\begin{aligned}G^{(j)}(x) &= \left(\sum_{i=0}^{j}\frac{k!}{(k-i)!}\,C_j^ix^{k-i}\right)e^x-\sum_{i=0}^{n-k}\frac{(i+k)!}{i!(i+k-j)!}\,x^{i+k-j}\\ &\quad+\frac{k(n+1)!}{(n-k+2)!(n-j+1)!}\,x^{n-j+1}-\frac{(n+2)!}{(n-k+2)!}\big(e^x-S_{n-j}(x)\big),\\ G^{(k+m)}(x) &= \sum_{i=0}^{k}\frac{k!}{(k-i)!}\,C_{k+m}^ix^{k-i}e^x-\sum_{i=m}^{n-k}\frac{(i+k)!}{i!(i-m)!}\,x^{i-m}\\ &\quad+\frac{k(n+1)!}{(n-k+2)!(n-k-m+1)!}\,x^{n-k-m+1}\\ &\quad-\frac{(n+2)!}{(n-k+2)!}\big(e^x-S_{n-k-m}(x)\big),\\ G^{(n+1)}(x) &= \sum_{i=0}^{k}\frac{k!}{(k-i)!}\,C_{n+1}^ix^{k-i}e^x+\frac{k(n+1)!}{(n-k+2)!}-\frac{(n+2)!}{(n-k+2)!}\,e^x,\\ G^{(n+2)}(x) &= \left(\sum_{i=0}^{k}\frac{k!}{(k-i)!}\,C_{n+1}^ix^{k-i}+\sum_{i=0}^{k-1}\frac{k!}{(k-i-1)!}\,C_{n+1}^ix^{k-i-1}\right.\\ &\quad\left.-\frac{(n+2)!}{(n-k+2)!}\right)e^x.\end{aligned}$$

Since $G^{(n+2)}(0) = 0$, $G^{(n+2)}(x)$ increases, hence $G^{(n+2)}(x) \geq 0$, $G^{(n+1)}(x)$ is increasing, $G^{(n+1)}(0) = 0$, $G^{(k+m)}(0) = 0$, $G^{(j)}(0) = 0$, and that $G(x) \geq 0$. QED

Lemma 4. *Let p, k and n be all positive integers such that $1 \leq p \leq k \leq n$. Then*

$$C_{n+1}^{k-1} \cdot C_{n+1}^{p} \geq (n+1) C_{n+1}^{k-p}. \tag{10}$$

Proof. We will use mathematical induction to prove the combinatorial inequality (10).

For $n \leq 2$, the inequality (10) apparently holds.

Suppose (10) is valid for $n = m$.

Let $n = m+1$. For $1 \leq p \leq k \leq m$, calculating easily results in

$$\begin{aligned} C_{m+2}^{k-1} \cdot C_{m+2}^{p} &= \frac{(m+2)^2}{(m-p+2)(m-k+3)} C_{m+1}^{k-1} \cdot C_{m+1}^{p} \\ &\geq \frac{(m+2)^2(m+1)}{(m-p+2)(m-k+3)} C_{m+1}^{k-p} \\ &= \frac{(m+2)(m+1)(m-k+p+2)}{(m-k+3)(m-p+2)} C_{m+2}^{k-p} \geq (m+2) C_{m+2}^{k-p}. \end{aligned}$$

For $1 \leq p \leq m, k = m+1$, one sees that

$$\begin{aligned} C_{m+2}^{m} \cdot C_{m+2}^{p} &\geq \frac{(m+2)^2(m+1)}{2(m-p+2)} C_{m+1}^{p} = \frac{(p+1)(m+1)(m+2)}{2(m-p+2)} C_{m+2}^{p+1} \\ &\geq (m+2) C_{m+2}^{p+1} = (m+2) C_{m+2}^{m-p+1}. \end{aligned}$$

For $p = k = m+1$ we get

$$C_{m+2}^{m} \cdot C_{m+2}^{m+1} = (m+2) C_{m+2}^{2} \geq (m+2)^2 > m+2 = (m+2) C_{m+2}^{0}.$$

Hence, for $n = m+1$, inequality (10) is sound. QED

Lemma 5. *For $n \geq k \geq 1$ and for $x \in [0, +\infty)$, we have*

$$\big(e^x - S_{n-k}(x)\big) x^k \leq \frac{k x^{n+1} e^x}{(n+1)(n-k+2)!} - \frac{n! - (n-k+2)(n+1)!}{(n-k+2)!} \big(e^x - S_n(x)\big). \tag{11}$$

Proof. Define a function $R : [0, +\infty) \to (-\infty, +\infty)$, such that

$$\begin{aligned} R(x) &= x^k e^x - x^k S_{n-k}(x) \\ &\quad + \frac{n! - (n-k+2)(n+1)!}{(n-k+2)!} \big(e^x - S_n(x)\big) - \frac{k x^{n+1} e^x}{(n+1)(n-k+2)!}. \end{aligned}$$

Calculating straightforwardly gives

$$R^{(j)}(x) = \sum_{i=0}^{j} \frac{k!}{(k-i)!} C_j^i x^{k-i} e^x - \sum_{i=0}^{n-k} \frac{(i+k)!}{i!(i+k-j)!} x^{i+k-j}$$

$$
\begin{aligned}
&+\frac{n!-(n-k+2)(n+1)!}{(n-k+2)!}\left(e^x-S_{n-j}(x)\right)\\
&-\frac{ke^x}{(n+1)(n-k+2)!}\sum_{i=0}^{j}\frac{(n+1)!}{(n-i+1)!}C_j^i x^{n-i+1},\\
R^{(k+m)}(x) \;=\;& \sum_{i=0}^{k}\frac{k!}{(k-i)!}C_{k+m}^i x^{k-i}e^x-\sum_{i=m}^{n-k}\frac{(i+k)!}{i!(i-m)!}x^{i-m}\\
&+\frac{n!-(n-k+2)(n+1)!}{(n-k+2)!}\left(e^x-S_{n-k-m}(x)\right)\\
&-\frac{ke^x}{(n+1)(n-k+2)!}\sum_{i=0}^{k+m}\frac{(n+1)!}{(n-i+1)!}C_{k+m}^i x^{n-i+1},\\
R^{(n+1)}(x) \;=\;& \frac{n!-(n-k+2)(n+1)!}{(n-k+2)!}e^x+\left(\sum_{i=0}^{k}\frac{k!}{(k-i)!}C_{n+1}^i\, x^{k-i}\right.\\
&\left.-\frac{k}{(n+1)(n-k+2)!}\sum_{i=0}^{n+1}\frac{(n+1)!}{(n-i+1)!}C_{n+1}^i\, x^{n-i+1}\right)e^x,
\end{aligned}
$$

where $1\le j\le k$, $1\le m\le n-k$, and $R^{(j)}(0)=0$, $R^{(k+m)}(0)=0$, $R^{(n+1)}(0)\le 0$.

Let

$$
\phi(x)=\sum_{i=0}^{k}\frac{k!}{(k-i)!}C_{n+1}^i x^{k-i}-\frac{k}{(n+1)(n-k+2)!}\sum_{i=0}^{n+1}\frac{(n+1)!}{(n-i+1)!}C_{n+1}^i x^{n-i+1},
$$

then, for $1\le p\le k$, using (10), by direct computation, one has

$$
\begin{aligned}
\phi^{(p)}(x) \;=\;& \sum_{i=0}^{k-p}\frac{k!}{(k-i-p)!}C_{n+1}^i x^{k-i-p}\\
&-\frac{k}{(n+1)(n-k+2)!}\sum_{i=0}^{n-p+1}\frac{(n+1)!}{(n-i-p+1)!}C_{n+1}^i x^{n-i-p+1},\\
\phi^{(p)}(0) \;=\;& \left(1-\frac{C_{n+1}^{k-1}\cdot C_{n+1}^{p}}{(n+1)C_{n+1}^{k-p}}\right)k!\,C_{n+1}^{k-p}\le 0.
\end{aligned}
$$

Since $\phi^{(k)}(x)$ decreases, then $\phi^{(p)}(x)\le 0$, thus $\phi(x)$ decreases, too. From $R^{(n+1)}(0)\le 0$ we deduce that $R^{(n+1)}(x)\le 0$, $R^{(n)}(x)$ is decreasing, hence $R^{(k+m)}(x)\le 0$, $R^{(j)}(x)\le 0$, $R(x)$ is decreasing, $R(x)\le 0$ owing to $R(0)=0$. QED

3. MAIN RESULT

Theorem 1. *Let b be a positive real number. For $x\in[0,b]$ we have*

$$
e^x\le S_n(x)+\alpha_n x^{n+1}, \tag{12}
$$

$$e^x \geq S_n(x) + \alpha_n x^{n+1} + \frac{(n+1)!\alpha_n - e^b}{b(n+1)(n+1)!}(b-x)x^{n+1} \tag{13}$$

where $S_n(x) = \sum_{k=0}^{n} x^k/k!$, $\alpha_{-1} = e^b$, $\alpha_n = (\alpha_{n-1} - \frac{1}{n!})/b$, $n \geq 0$, *and the coefficients* α_n *and* $\frac{(n+1)!\alpha_n - e^b}{(n+1)!(n+1)b}$ *are sharp in the sense that they could not be replaced by smaller and larger constants in* (12) *and* (13), *respectively.*

Proof. Let $F(x) = e^x - S_n(x) - \alpha_n x^{n+1} + \gamma(b-x)x^{n+1}$, $x \in [0,b]$, where $\gamma \geq 0$ is an undetermined coefficient. By a straightforward computation, it follows that

$$\alpha_n = [e^b - S_n(b)]/b^{n+1},\ F(0) = F(b) = 0,$$

$$\begin{aligned}
F^{(k)}(x) &= e^x - S_{n-k}(x) - \frac{(n+1)!}{(n-k+1)!}\alpha_n x^{n-k+1} + \gamma\Big(\frac{(n+1)!}{(n-k+1)!}b \\
&\quad - \frac{(n+2)!}{(n-k+2)!}x\Big)x^{n-k+2},\ F^{(k)}(0) = 0,\ 0 \leq k \leq n, \\
F^{(k)}(b) &= e^b - S_{n-k}(b) - \frac{(n+1)!}{(n-k+1)!}\alpha_n b^{n-k+1} - \frac{k(n+1)!}{(n-k+2)!}\gamma\, b^{n-k+2}, \\
F^{(n+1)}(x) &= e^x + \gamma\big((n+1)!b - (n+2)!x\big) - (n+1)!\alpha_n, \\
F^{(n+1)}(0) &= 1 - (n+1)!\alpha_n + b\gamma(n+1)!, \\
F^{(n+1)}(b) &= e^b - (n+1)!\alpha_n - b\gamma(n+1)(n+1)!, \\
F^{(n+2)}(x) &= e^x - \gamma(n+2)!,\ F^{(n+2)}(0) = 1 - \gamma(n+2)!, \\
F^{(n+2)}(b) &= e^b - \gamma(n+2)!
\end{aligned}$$

Clearly, $F^{(n+2)}(x)$ is monotonically increasing.

3.1. If $\frac{1}{(n+2)!} < \gamma < \frac{e^b}{(n+2)!}$, then $F^{(n+2)}(0) < 0$, $F^{(n+2)}(b) > 0$, therefore $F^{(n+2)}(x)$ has an only one zero, which is a minimum of $F^{(n+1)}(x)$ on $(0,b)$.

3.1.1 If $\gamma \leq \frac{(n+1)!\alpha_n - 1}{(n+1)!b}$ and $\gamma < \frac{e^b - (n+1)!\alpha_n}{(n+1)(n+1)!b}$, then $F^{(n+1)}(0) \leq 0$, $F^{(n+1)}(b) > 0$, thus $F^{(n+1)}(x)$ has a unique zero on $(0,b)$, therefore $F^{(n)}(x)$ has a unique minimum on $(0,b)$. If $\frac{1}{(n+2)!} < \gamma \leq \big((n+1)!\alpha_n - 1\big)/(n+1)!b$, from (7) and (8), $F^{(n)}(x)$ has a unique minimum on $(0,b)$. From (9) it follows that $F^{(k)}(b) > 0$, $F^{(k)}(x)$ has only one minimum. Hence $F(x) \leq 0, x \in [0,b]$. The proof of (12) is completed.

3.1.2 If $e^b/(n+2)! > \gamma \geq \big(e^b - (n+1)!\alpha_n\big)/(n+1)(n+1)!b$, then $F^{(n+1)}(0) > 0$, $F^{(n+1)}(b) \leq 0$, wherefore $F^{(n+1)}(x)$ has an only zero on $(0,b)$, $F^{(n)}(x)$ has a unique maximum on $(0,b)$. From (11) we have $F^{(k)}(b) < 0$ and $F^{(k)}(x)$ has only one maximum on $(0,b)$. Hence $F(x) \geq 0$, $x \in [0,b]$. The proof of (13) is completed.

3.2. If $\gamma \leq 1/(n+2)!$, then $F^{(n+2)}(x) \geq 0$, $F^{(n+1)}(x)$ increases. Also $F^{(n+1)}(0) < 0$, $F^{(n+1)}(b) > 0$. Going through the same process as (3.1.1) again, then (12) is sound.

3.3. If $\gamma \geq e^b/(n+2)!$, it is seen easily that $F^{(n+2)}(x) \leq 0$, $F^{(n+1)}(x)$ decreases. Then $F^{(n+1)}(0) > 0$, $F^{(n+1)}(b) < 0$. Also going through the same process as (3.1.2) again, (13) is valid.

Indeed, the coefficients α_n and $\big((n+1)!\alpha_n - e^b\big)/\big((n+1)!b(n+1)\big)$ of inequalities (12) and (13) cannot be replaced by smaller and larger constants, respectively. In this sense the inequalities (12) and (13) are sharp. QED

REMARK 1. A similar argument yields

Theorem 2. *For a given constant $b>0$, we have*

$$e^x \geq 1+\alpha_0 x+(b-x)\sum_{k=1}^{n}\beta_k x^k \qquad (x\in[0,b]) \tag{14}$$

$$e^x \leq 1+\alpha_0 x+(b-x)\sum_{k=1}^{n}\beta_k x^k+\frac{e^b+(n+1)\beta_n}{(n+1)!(n+1)b}(b-x)x^{n+1}, \tag{15}$$

where $\beta_1=\alpha_1, \beta_{k+1}=\big(\beta_k+1/(k+1)!\big)/b$.

4. AN APPLICATION OF THE INEQUALITY (12)

Integrating on both sides of (12) over $[0,b]$ establishes

$$\int_0^b e^x\,\mathrm{d}x \leq \int_0^b S_n(x)\,\mathrm{d}x+\int_0^b \alpha_n x^{n+1}\,\mathrm{d}x,$$

$$e^b \leq S_{n+1}(b)+b^{n+2}\alpha_n/(n+2). \tag{16}$$

Using $\alpha_n=\big(e^b-S_n(b)\big)/b^{n+1}$ and rearranging (16) results in

$$e^b-S_{n+1}(b)\leq b^{n+2}/(n-b+2)(n+1)!,\ b<n+2$$

that is

$$e^x-S_{n+1}(x)\leq x^{n+2}/(n-x+2)(n+1)! \qquad (x\in[0,n+2)).$$

Hence we get inequality (5) by integrating (12). In this sense the inequality (12) is better than (5).

REMARK 2. Analogously, the inequality (13) produces

$$e^x-S_n(x)\geq\frac{(n+1)(n+2)(n+3)-xe^x}{(n+1)(n+3)-(n+2)x}\cdot\frac{x^{n+1}}{(n+2)!},\ x\in\left[0,\frac{(n+1)(n+3)}{n+2}\right).$$

REMARK 3. Recurring β_k directly reduces to

$$\beta_k=\big(e^b+S_k(b)-2(1+b)\big)/b^{k+1},\ b>0. \tag{17}$$

Then, in the same way as remark 2, inequalities (14) and (15) imply

$$\frac{e^x-1}{x}\geq\frac{1}{2}(1+e^x)+\sum_{k=1}^{n}\frac{e^x+S_k(x)-2(1+x)}{(k+1)(k+2)} \tag{18}$$

and

$$\frac{e^x-1}{x} \le \frac{1}{2}(1+e^x) + \sum_{k=1}^{n} \frac{e^x + S_k(x) - 2(1+x)}{(k+1)(k+2)} + \frac{x^{n+1}e^x + (n+1)\big(e^x + S_n(x) - 2(1+x)\big)}{(n+3)!(n+1)(n+2)} \tag{19}$$

for all $x \in (0, +\infty)$.

REMARK 4. In fact, inequality (6) is also a simple consequence of the monotonicity for $\big(e^x - S_n(x)\big)/x^{n+1}$, $x \in (0, +\infty)$. This provides another proof of inequality (5).

Acknowledgments. The author is indebted to the referee for many helpful suggestions and for valuable additions to the list of references.

REFERENCES

1. G. H. HARDY, J. E. LITTLEWOOD, G. POLYA: *Inequalities.* Cambridge University Press, 2nd Edition, 1952.
2. E. F. BECKENBACH, ED.: *General Inequalities* II. Proceedings of 2nd International Conferences on General Inequalities, Birkhaeuser, Basel, 1980.
3. GUO BAINI, QI FENG: *Improvement for the Upper Bound of a Class of Elliptic Integral.* J. Jiaozuo Mining Institute, Vol. 14, No. 6 (1995), 125–128.
4. KUANG JICHANG: *Applied Inequalities*, 2nd edition. Hunan Education Press, Changsha, China, 1993.
5. M. MERKLE: *Inequalities for Residuals of Power Series: a Review.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. **6** (1995), 79–85.
6. D. S. MITRINOVIC: *Analytic Inequalities.* Springer-Verlag, 1970.
7. QI FENG: *Extensions and Sharpenings of Jordan's and Kober's Inequality.* J. Mathematics for Technology, Vol. 12, No. 4 (1996), 70–74.
8. QI FENG, GUO BAINI: *Estimate for Upper Bound of an Elliptic Integral.* Mathematics in Practice and Theory, Vol. 26, No. 3 (1996), 285–288.
9. QI FENG, GUO BAINI: *The Estimation of Inferior Bound for an Ellipse Integral.* J. Mathematics for Technology, Vol. 10, No. 1 (1994), 87–90.

(Received April 26, 1996)

Department of Mathematics,
University of Science and Technology of China,
Hefei 230026, Anhui, China; and
Department of Mathematics,
Jiaozuo Institute of Technology,
Jiaozuo 454000, Henan, China